

Sheaves of modules:

Throughout, X will denote a noetherian scheme and \mathcal{O}_X will denote its structure sheaf. For a more general setting, see EGA (Elements de Géométrie algébrique) Stacks project: online source

(1) A sheaf of \mathcal{O}_X -modules is a sheaf \mathcal{F} on X s.t.
 \forall open $U \subset X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$ -module
s.t. \forall open sets $V \subset U \subset X$ and $a \in \mathcal{O}_X(U)$,
 $s \in \mathcal{F}(U)$, we have $(as)|_V = a|_V \cdot s|_V$

(2) A morphism of \mathcal{O}_X -modules $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves s.t. \forall open $U \subset X$, $\varphi(U): \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$ -modules.

(3) The kernel, cokernel and image of a morphism of \mathcal{O}_X -modules is an \mathcal{O}_X -module. The quotient of two \mathcal{O}_X -modules is an \mathcal{O}_X -module: cokernel of an inclusion.

(4) The direct sums, direct products, direct limits and inverse limits of \mathcal{O}_X -modules are again \mathcal{O}_X -modules.
(see exercises in Chapter II. Section 1)

(5) Tensor products: Given \mathcal{O}_X -modules \mathcal{F}, \mathcal{G} , the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheaf associated to the presheaf

$$U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$$

(this presheaf is seldom a sheaf)

(6) The sheaf Hom: Given \mathcal{F}, \mathcal{G} , the set

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) := \left\{ \begin{array}{l} \text{morphisms of } \mathcal{O}_X\text{-modules} \\ \text{from } \mathcal{F} \text{ to } \mathcal{G} \end{array} \right\}$$

is a module over $\mathcal{O}_X(X)$.

the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the sheaf

$$U \longmapsto \text{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$$

which is also an \mathcal{O}_X -module.

(7) An \mathcal{O}_X -module \mathcal{F} is quasi-coherent if there exists an open cover of X by affine open subschemes $U = \text{Spec } A$ s.t. \exists an A -module M with $\mathcal{F}|_U \cong \tilde{M}$

recall: \tilde{M} is the sheaf on $\text{Spec } A$ s.t. \forall basic open $\text{Spec } A[f^{-1}]$, $\tilde{M}(\text{Spec } A[f^{-1}]) = M[f^{-1}]$.

One can prove that then \forall open affine

$$V = \text{Spec } B \subset X, \exists B\text{-module } N \text{ s.t. } \mathcal{F}|_V \cong \tilde{N}.$$

A quasi-coherent sheaf is called coherent if, in

addition, the modules M are finite.

(8) An \mathcal{O}_X -module is called free if it is isomorphic, as an \mathcal{O}_X -module, to a direct sum of sheaves isomorphic to \mathcal{O}_X . It is locally free if X has an open covering by open sets U s.t. $\mathcal{F}|_U$ is free.

Note: Locally free sheaves are quasi-coherent:

$$\mathcal{O}_X^{\oplus I} |_{\text{Spec } A} = \widetilde{(A^{\oplus I})}$$

An isomorphism $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus I}$ is called a trivialization of \mathcal{F} on U .

The rank of a locally free sheaf is the number of copies of \mathcal{O}_U in any trivialization, it can be finite or infinite. The rank of a locally free sheaf is constant on each connected component of X .

Locally free sheaves are coherent when they have finite rank.

gluing data: Suppose \mathcal{F} is locally free of rank n .

$$\exists X = \bigcup_{i=1}^m U_i$$

s.t. $\forall i$

$\exists i$

$$\mathcal{F}|_{U_i} \xrightarrow{\cong \varphi_i} (\mathcal{O}_{U_i})^{\oplus n}$$

$$\begin{array}{ccc}
 \mathcal{O}_U \otimes \mathcal{O}(U_i \cap U_j) & \xleftarrow{\varphi_i} & \mathcal{O}_U \otimes \mathcal{O}(U_i) \\
 \downarrow \varphi_j \circ \varphi_i^{-1} & \cong & \downarrow \text{Id} \\
 \mathcal{O}_U \otimes \mathcal{O}(U_i \cap U_j) & \xleftarrow{\varphi_j} & \mathcal{O}_U \otimes \mathcal{O}(U_j)
 \end{array}$$

def: the gluing data is $\left\{ \varphi_j \circ \varphi_i^{-1} : \mathcal{O}_U \otimes \mathcal{O}(U_i \cap U_j) \right\}$

$$\varphi_j \circ \varphi_i^{-1} \longleftrightarrow \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \in GL_{\mathcal{O}_U \otimes \mathcal{O}(U_i \cap U_j)}(\mathcal{O}_U \otimes \mathcal{O}(U_i \cap U_j))$$

$$a_{ij} \in \mathcal{O}_X(U_i \cap U_j)$$

the matrix (a_{kl}) is called a transition matrix.

(9) An invertible sheaf \mathcal{L} is a locally free sheaf of rank 1.

Claim: If we put $\mathcal{L}^{-1} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$,

then there is a natural isomorphism $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \cong \mathcal{O}_X$.

More generally, for any \mathcal{O}_X -module, \exists a homomorphism

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \longrightarrow \mathcal{O}_X$$

on any U :

$$\mathcal{F}(U) \otimes_{\mathcal{O}_U} \mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}(U), \mathcal{O}_U) \longrightarrow \mathcal{O}_X(U)$$

$$s \otimes l \longmapsto l(s)$$

defines a morphism of presheaves, it factors through

the sheaf $\mathcal{F} \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$

Notation: People often write $\mathcal{F}^* := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

When \mathcal{F} is locally free of rank 1, one shows that this

homomorphism is an isomorphism of \mathcal{O}_X -modules.

In fact, one can show that $\otimes_{\mathcal{O}_X}$ defines a group structure on the set $\{\text{invertible sheaves}\} / \sim$: this is the Picard group of X , denoted $\text{Pic}(X)$. (abelian)

Important example: Fix an algebraically closed field k .

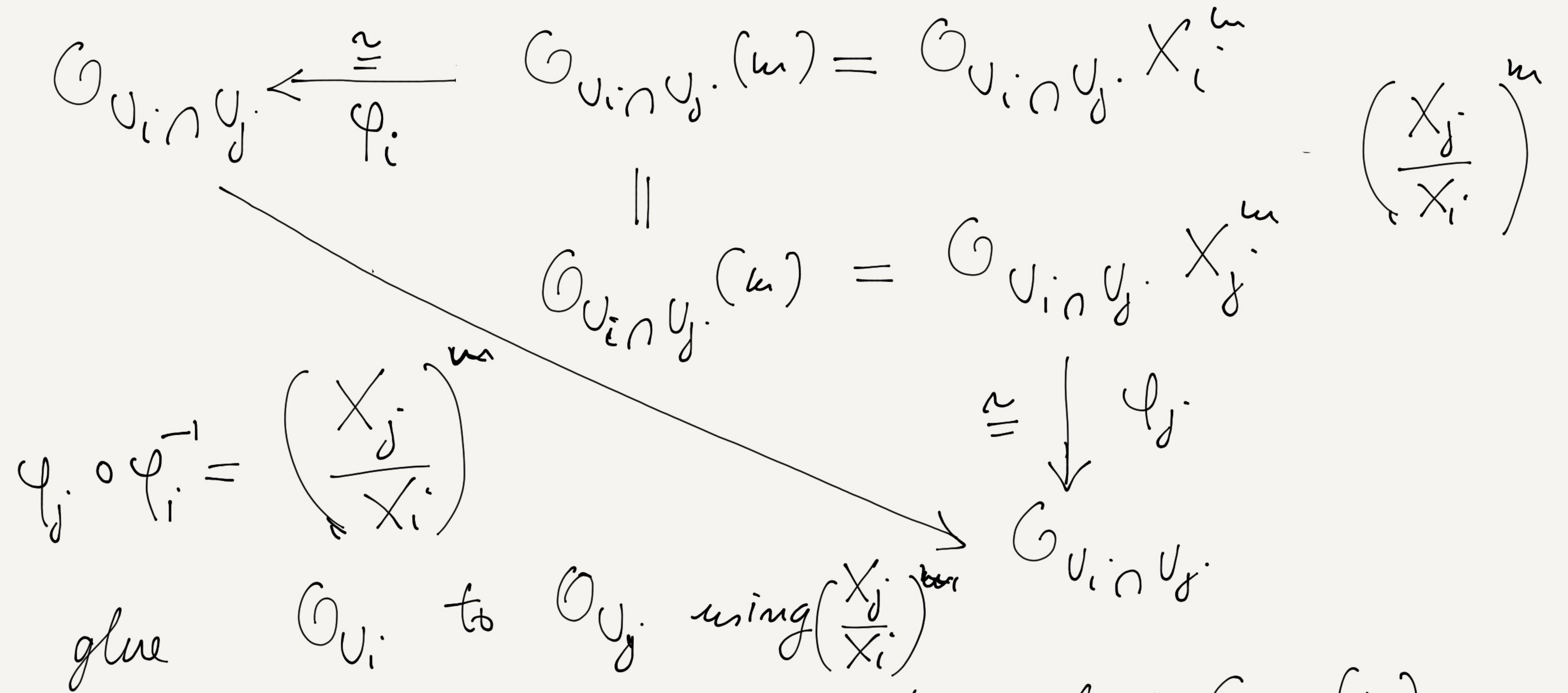
$$X := \mathbb{P}_k^n = \text{Proj } k[X_0, \dots, X_n]$$

For any $m \in \mathbb{Z}$, we define an invertible sheaf $\mathcal{O}_{\mathbb{P}^n}(m)$.

$$\mathbb{P}_k^n := \bigcup_{i=0}^n U_i \quad U_i = \text{Spec } k\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right]$$
$$= \{p \mid p \neq X_i\}$$

On U_i , define $\mathcal{O}_{U_i}(m) := \mathcal{O}_{U_i} \underbrace{X_i^m}_{\text{formal generator}}$

we glue $\mathcal{O}_{U_i}(u)$ with $\mathcal{O}_{U_j}(u)$ on $U_i \cap U_j$:



\Rightarrow well-defined sheaf on \mathbb{P}^n , called $\mathcal{O}_{\mathbb{P}^n}(u)$.

Exercise: $\mathcal{O}_{\mathbb{P}^n}(-u) \cong \mathcal{O}_{\mathbb{P}^n}(u)^{-1} := \text{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{\mathbb{P}^n}(u), \mathcal{O}_{\mathbb{P}^n})$

In general, if \mathcal{L} has trivializations $\mathcal{L}|_{U_i} \xrightarrow{\cong} \mathcal{O}_{U_i}$ with transition functions $f_{ij} := \varphi_j \circ \varphi_i^{-1}$ on $U_i \cap U_j$, then \mathcal{L}^{-1} has trivializations $\mathcal{L}^{-1}|_{U_i} \xrightarrow[\varphi_i]{\cong} \mathcal{O}_{U_i}$ with transition functions $g_{ij} := (f_{ij})^{-1} \in \mathcal{O}_{U_i \cap U_j}$.

Claim: $\left\{ \mathcal{O}_{\mathbb{P}^n}(u), u \in \mathbb{Z} \right\} / \sim \xleftarrow{\cong} \mathbb{Z}$

This will follow from:

Claim: $\Gamma(\mathcal{O}_{\mathbb{P}^n}(u)) \cong \left\{ \begin{array}{l} \text{homogeneous polynomials of} \\ \text{degree } u \end{array} \right\} \cup \{0\}$

Proof: Choose $s \in \Gamma(\mathcal{O}_{\mathbb{P}^n}(u)) = \mathcal{O}_{\mathbb{P}^n}(u)(\mathbb{P}^n)$
 $\forall i \quad s|_{U_i} \in \Gamma(\mathcal{O}_{\mathbb{P}^n}(u)|_{U_i}) = \Gamma(\mathcal{O}_{U_i} X_i^u)$

$$s|_{U_i} \in k\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right] X_i^m$$

$$\exists f_i\left(\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right) \text{ s.t. } s|_{U_i} = f_i\left(\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right) X_i^m$$

$s|_{U_i}$ has to glue to $s|_{U_j}$ on $U_i \cap U_j$.

$$\Rightarrow f_i\left(\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right) X_i^m = f_j\left(\frac{X_0}{X_j}, \dots, \frac{X_n}{X_j}\right) X_j^m$$

\Rightarrow $f_i X_i^m$ and $f_j X_j^m$ have no denominators.
 $\underbrace{f_i X_i^m}_{=: P_i}$ $\underbrace{f_j X_j^m}_{=: P_j}$
 Polynomial and $P_i = P_j \quad \forall i, j$

Put $P := P_i = P_j$
 define a map $T: (\mathcal{O}_{\mathbb{P}^n}(k))_S \longrightarrow S_m \subset S = k[X_0, \dots, X_n]$
 $s \longmapsto P \cdot s$

To show this map is an isomorphism, we show it has an inverse:

Start with $P \in S_n$, $\forall i$, we can

write $P = \sum_i \left(\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i} \right) X_i^k$

\rightarrow factor X_i^k
 \downarrow
 a section s_i of $\mathcal{O}_{U_i}(k)$

$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ by def. \Rightarrow we get an
 inverse map $S_n \rightarrow \Gamma(\mathcal{O}_{\mathbb{P}^n}(k))$.

\square