Recall: Theorem: X irreducible, separated, of finite type over an algebraically closed field k. Then $\Omega^1_{X/k}$ is a locally free sheaf of rank $= \dim X$ iff X is a nonsingular variety $/k$.

Reason: $\forall x$ closed $\quad \Omega^1_{X/k,x} \otimes_{\mathcal{O}_{X,x}} k \cong m_x/m_x^2$

Recall: If $\begin{array}{c} Y \subset X \\ \downarrow \\ S \end{array}$ is a closed subscheme, then we have the natural exact sequence

pull-back via inclusion

$$\mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega^1_{X/S}\big|_Y \longrightarrow \Omega^1_{Y/S} \longrightarrow 0$$

Theorem (8.17) If $S = \operatorname{Spec} k$ k alg. closed and X is a nonsingular variety, then Y is nonsingular iff

(1) $\Omega^1_{Y/k}$ is locally free

(2) the natural sequence

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega^1_{X/k}\big|_Y \longrightarrow \Omega^1_{Y/k} \longrightarrow 0$$

is exact.

Furthermore, if the above holds, then $\mathcal{I}$ is locally generated by $d = \operatorname{codim}_X(Y)$ elements and $\mathcal{I}/\mathcal{I}^2$ is locally free of rank d.

X variety $/k$

Def: The tangent sheaf of X is

$$\mathcal{T}_X := \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathcal{O}_X)$$

note: $\forall x \in X$ closed point, $\mathcal{T}_{X,x} \otimes_{\mathcal{O}_{X,x}} k \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^* = T_x X$

If $Y \subset X$ is a closed subscheme, the normal sheaf of Y in X is $\mathcal{N}_{Y/X} := (\mathcal{I}/\mathcal{I}^2)^* := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$

The sheaf $\mathcal{I}/\mathcal{I}^2$ is called the conormal sheaf.

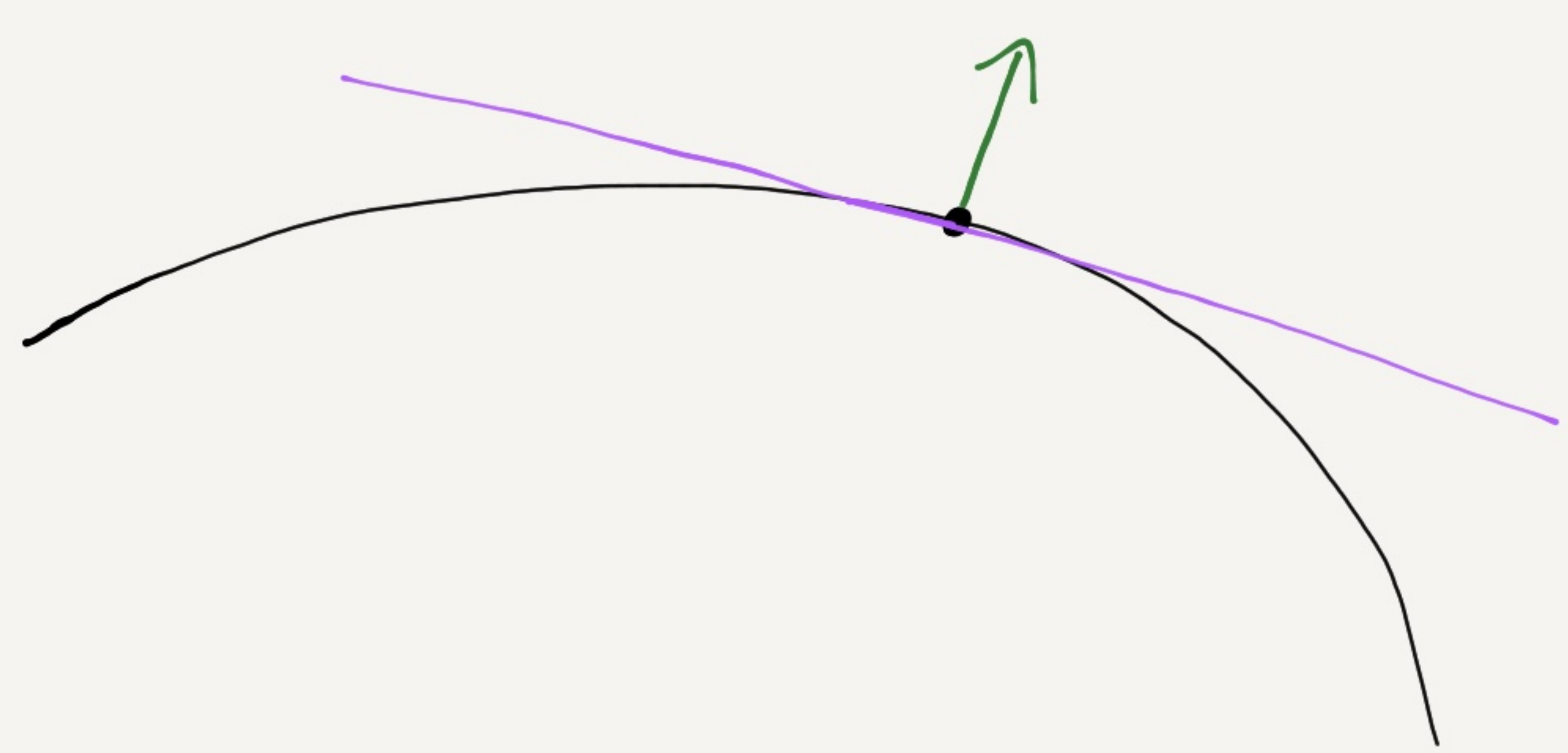
If Y is also a nonsingular variety, then we can dualize the exact sequence

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega^1_X|_Y \longrightarrow \Omega^1_Y \longrightarrow 0 \qquad /k$$

to obtain

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_X|_Y \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0$$

on closed points: $0 \to T_x Y \to T_x X \longrightarrow N_{Y/X,x} \longrightarrow 0$



Def: If $Y \subset X$ is a closed subscheme of a nonsingular variety, then we say Y is a local complete intersection in X if the ideal sheaf $\mathcal{I}_Y$ can be locally generated by $d = \mathrm{codim}_X(Y)$ elements.

Prop.: If $Y \subset X$ is a local complete intersection, (l.c.i.) then $\mathcal{I}_Y / \mathcal{I}_Y^2$ is locally free of rank d and the sequence

$$0 \to \mathcal{I}_Y / \mathcal{I}_Y^2 \to \Omega^1_X|_Y \to \Omega^1_Y \to 0$$

is exact.

$$\left(\Rightarrow \Lambda^n \Omega^1_X|_Y \cong \Lambda^{n-d} \Omega^1_Y \otimes \Lambda^d \left(\mathcal{I}_Y / \mathcal{I}_Y^2 \right) \right)$$

Def: X nonsingular variety$/k$

The canonical sheaf of X is

$$K_X := \Lambda^n \Omega^1_X \quad \text{where } n = \dim X.$$

Note: When $Y \subset X$ is nonsingular, then

$$K_Y \cong K_X \otimes \Lambda^d \mathcal{N}_{Y/X}$$

Note: Being l.c.i. is independent of the ambient scheme X.

Cohen-Macaulay is a generalization of l.c.i.

Let A be a ring, M an A-module

Def: A sequence $a_1, \dots, a_n$ of elements of A is called M-regular if a_1 is not a zero divisor in M, i.e., multiplication by a_1 : $M \xrightarrow{a_1} M$ is injective, a_2 is not a zero divisor in M/a_1M, a_3 is not a zero divisor in $M/a_1M + a_2M$, ...

If A is a local ring with maximal ideal $\mathfrak{m}$, the depth of M is the maximum length of a regular sequence of elements of $\mathfrak{m}$.

We say that A is a Cohen-Macaulay ring if

$$\operatorname{depth} A = \dim A.$$

Note: We always have depth $\leq$ dim.

Facts: (1) Regular local rings are Cohen-Macaulay, quotients of Cohen-Macaulay rings by regular sequences are Cohen-Macaulay. In particular, local rings of l.c.i. schemes are Cohen-Macaulay.

(2) Noetherian local rings of dim 0 (i.e., local Artinian rings) are Cohen-Macaulay.

(3) One-dimensional reduced noetherian local rings are Cohen-Macaulay.

(4) Two-dimensional integrally closed noetherian local domains are Cohen-Macaulay.

(5) If A is a finitely generated Cohen-Macaulay algebra over a field k with an action of a finite group G, then the subring $A^G \hookrightarrow A$ of G-invariants is Cohen-Macaulay.

(6) Determinantal rings are Cohen-Macaulay:

For any ring A, given an $m \times n$ matrix $(a_{ij}) =: M$ with entries in A, consider the ideal I generated by the minors of size $r \times r$ of M: this is a determinantal ideal.

A ring is called determinantal if it is the quotient of a regular local ring by a determinantal ideal.

Theorem (8.22A Serre):

A noetherian ring A is normal (every localization of A at a prime ideal is an integrally closed domain) iff it satisfies

$R_1 \to$ (1) $\forall$ prime $p \subset A$ of height ≤ 1, A_p is regular

$S_2 \to$ (2) $\forall$ prime $p \subset A$ of height ≥ 2, $\mathrm{depth}(A_p) \geq 2$.

$\Rightarrow$ Prop: (8.23) If $Y \subset X$ is a l.c.i. $/k$. Then

(1) Y is Cohen-Macaulay

(2) Y is normal iff it is regular in codim. 1. (R_1)

If a scheme is Cohen-Macaulay, S_2 is automatically satisfied.