Consider the subset

$$B := \left\{ (x, H) \;\middle|\; \begin{array}{l} x \in H, \text{ either } x \text{ is a singular} \\ \text{point of } X \cap H \text{ or } H \supset X \end{array} \right\}$$

$$\subset X \times_k |\mathcal{O}_{\mathbb{P}^n}(1)|$$

<u>Claim:</u> B has a natural structure of a closed subscheme of $X \times_k |\mathcal{O}_{\mathbb{P}^n}(1)|$.

<u>Proof:</u> If $(x, H) \in B$, let $f \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ be an equation for H, i.e., $Z(f) = H$ and let $\{P_1, \dots, P_r\}$ be a set of generators for I_X. Then, $I_{X \cap H}$ is generated by $(f, P_1, \dots, P_r)$.

We have $(x, H) \in B \iff$ the Zariski tangent space $T_x X \cap H$ has dimension $> d-1$ where $d = \dim X$

By the description of the embedded tangent space to $X \cap H$, $T_x X \cap H$ has dim $> d-1$

$$\iff \left(\frac{\partial f}{\partial X_i}(x), \frac{\partial P_j}{\partial X_i}(x) \right)_{\substack{1 \leq j \leq r \\ 0 \leq i \leq n}} \text{ has rank} < n-d+1$$

So the minors of size $(n-d+1) \times (n-d+1)$ of the above matrix give polynomial equations for B as a closed subscheme of $X \times |\mathcal{O}_{\mathbb{P}^n}(1)|$ in the coordinates

$(X_0, \dots, X_n)$ on $X \subset \mathbb{P}^n$ and coordinates

$(\lambda_0, \dots, \lambda_n)$ on $|\mathcal{O}_{\mathbb{P}^n}(1)|$ where $f = \sum_{i=0}^{n} \lambda_i X_i$

$$= (\mathbb{P}^n)^* \cong \mathbb{P}^n$$

(abstract)

$\square$

We continue the proof of Bertini's theorem:

Choose $(x, H) \in X \times |\mathcal{O}_{\mathbb{P}^n}(1)|$, $H = Z(f)$

Let i be such that $x \in U_i = D_+(X_i)$

On U_i, the ideal $I_{H \cap U_i}$ is generated by $\dfrac{f}{X_i}$

consider the linear map

$$\varepsilon_x : \Gamma(\mathcal{O}_{\mathbb{P}^n}(1)) \longrightarrow \overbrace{\mathcal{O}_{X,x}/\mathfrak{m}_x^2}^{\dim.\ d+1/k} \supset \mathfrak{m}_x/\mathfrak{m}_x^2$$

$$f \longmapsto (f/X_i)_x \text{ modulo } \mathfrak{m}_x^2$$

We have $H \supset X$ or x is a singular point of $X \cap H$

$\iff f \in \ker \varepsilon_x$.

Because $(f/X_i)_x \in \mathcal{O}_{X,x}$ generates the ideal of $H \cap X$ in $\mathcal{O}_{X,x}$

$H \supset X \iff (f/X_i)_x = 0$ in $\mathcal{O}_{X,x}$ = localization of ring of $X \cap U_i$

(localization map is injective because X is integral.)

($H \not\supset X$ and x is a non singular point of $X \cap H$)

$\iff T_x(X \cap H)$ has dim. $d-1$.

$\iff \mathfrak{m}_{X \cap H, x} / \mathfrak{m}^2_{X \cap H, x}$ has dim. $d-1$

<u>Note that:</u> $\mathcal{O}_{X\cap H, x} = \mathcal{O}_{X,x} \big/ \left(f/X_i\right)_x$

and $\mathfrak{m}_{X\cap H, x} = \mathfrak{m}_{X,x} \big/ \left(f/X_i\right)_x$

$\left(x \in X \cap H \iff \left(f/X_i\right)_x \in \mathfrak{m}_{X,x}\right)$

and $\mathfrak{m}_{X\cap H,x} \big/ \mathfrak{m}^2_{X\cap H,x} = \dfrac{\mathfrak{m}_{X,x} \big/ \mathfrak{m}^2_{X,x}}{\left(\left(f/X_i\right)_x \bmod \mathfrak{m}^2_{X,x}\right)}$

So $\dim \mathfrak{m}_{X\cap H,x} \big/ \mathfrak{m}^2_{X\cap H,x} = d-1$

$\iff \left(f/X_i\right)_x \neq 0$ modulo $\mathfrak{m}^2_{X,x}$

So, if $x \in X \cap H$ is a singular point, then

$\left(f/X_i\right)_x = 0$ modulo $m_{X,x}^2$.

This proves that $(x, H) \in B \Leftrightarrow f \in \operatorname{Ker} \varepsilon_x$.

<u>Note</u>: that if we change the index i s.t. $x \in U_i$, ε_x changes by multiplication by $\left(\frac{X_i}{X_j}\right)(x)$ which is a nonzero scalar. So $\operatorname{Ker} \varepsilon_x$ is independent of i.

<u>Note</u>: ε_x is surjective because m_x is generated by images (in $\mathcal{O}_{X,x}$) of some degree 1 polynomials, i.e., elements of $\Gamma(\mathcal{O}_{\mathbb{P}^n}(1))$ (we are over an alg. closed field).

$$m_{\mathbb{P}^n,x} \twoheadrightarrow m_{X,x}$$

If $x = (a_0, \dots, a_n)$ $\quad \mathfrak{m}_{\mathbb{P}^n, x} = (a_i X_j - a_j X_i)_{0 \le i, j \le n}$

So $\operatorname{Ker} \varepsilon_x$ has dim. $n+1-(d+1) = n-d$ over k.

Now:

$$
\begin{tikzcd}
q_1^{-1}(x) \arrow[r, phantom, "\subset"] \arrow[d] \arrow[dr, phantom, "\square"] & B \arrow[r, phantom, "\subset"] \arrow[d, "q_1"] \arrow[rrd, bend left, "q_2"] & X \times_k |\mathcal{O}_{\mathbb{P}^n}(1)| \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\
x \arrow[r, phantom, "\in"] & X & & (\mathbb{P}^n)^* = |\mathcal{O}_{\mathbb{P}^n}(1)| \cong \mathbb{P}^n
\end{tikzcd}
$$

We show $q_2(B)$ is a proper closed subset of $|\mathcal{O}_{\mathbb{P}^n}(1)|$

By Ex. II.3.22 about the fibers of a morphism, the underlying set of $q_1^{-1}(x)$ is the set of preimages of x by q_1.

$$
\begin{aligned}
q_1^{-1}(x) &= \{ (x, H) \mid (x, H) \in B \} \\
&= \{ (x, H) \mid f \in \operatorname{Ker} \varepsilon_x \} = \{x\} \times \mathbb{P} \operatorname{Ker} \varepsilon_x
\end{aligned}
$$

So $\dim q_1^{-1}(x) = \dim(\{x\} \times \mathbb{P}\operatorname{Ker}\varepsilon_x) = \dim(\mathbb{P}\operatorname{Ker}\varepsilon_x)$
$= n-d-1$

Ex. II.3.22 : $\dim B = \dim q_1(B) + \dim q_1^{-1}(x)$
$\leq \dim X + n-d-1 = n-1$

Ex. II.4.4 the image of a proper scheme is proper.

B is a closed subscheme of $X \times_k |\mathcal{O}_{\mathbb{P}^n}(1)|$, hence it is proper$/k$. So $q_2(B)$ is also proper, hence it is closed in $|\mathcal{O}_{\mathbb{P}^n}(1)|$, and $\dim q_2(B) \leq \dim B \leq n-1$.

Since $\dim \mathbb{P}^n = n$, $q_2(B) \subsetneq |\mathcal{O}_{\mathbb{P}^n}(1)|$.

$\square$