Consider the subset

\[ B := \{ (x, H) \mid x \in H, \text{ either } x \text{ is a singular point of } X \cap H \text{ or } H \supset X \} \]

\[ C = \frac{X \times X}{\mathbb{P}^n(1)} \]

**Claim:** \( B \) has a natural structure of a closed subscheme of \( \frac{X \times X}{\mathbb{P}^n(1)} \).

**Proof:** If \((x, H) \in B\), let \( f \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))\) be an equation for \( H \), i.e., \( Z(f) = H \) and let \( \{P_1, \ldots, P_n\} \) be a set of generators for \( I_X \). Then, \( I_{X \cap H} \) is generated by \((f, P_1, \ldots, P_n)\).
We have \((x, H) \in \mathcal{B} \iff\) the Zariski tangent space \(T_x X \cap H\) has dimension \(> d - 1\) where \(d = \dim X\).

By the description of the embedded tangent space to \(X \cap H\), \(T_x X \cap H\) has \(\dim > d - 1\).

\[
\begin{pmatrix}
\frac{\partial f_i}{\partial x_i}(x), & \frac{\partial P_j}{\partial x_i}(x)
\end{pmatrix}_{0 \leq i \leq n, \ 0 \leq j \leq n}
\]

has rank \(< n - d + 1\).

So the minors of size \((n-d+1) \times (n-d+1)\) of the above matrix give polynomial equations for \(\mathcal{B}\) as a closed subscheme of \(X \times \mathbb{A}^{n-p_a(1)}\) in the coordinates.
We continue the proof of Bertini's theorem:

Choose \((x, H) \in X \times |\mathcal{O}_{\mathbb{P}^n}(r)|\) \(\text{ and } H = Z(F)\)

Let \(i\) be such that \(x \in U_i = D + (X_i)\)

On \(U_i\), the ideal \(I_{H \cap U_i}\) is generated by \(\frac{f}{X_i}\)

Consider the linear map

\[
\varepsilon_x : \Gamma(\mathcal{O}_{\mathbb{P}^n}(1)) \to \frac{X_i x}{x^2} \to \frac{1}{(\frac{f}{X_i})} \mod \mu^2
\]
We have \( H \supseteq X \ni x \) is a singular point of \( X \cup H \).

\[ \implies \quad f \in \ker \mathfrak{m}_x \]

Because \( (\sqrt{x_i})_x \in O_{X,x} \) generates the ideal of \( H \supseteq X \) in \( O_{X,x} \),

\[ H \supseteq X \implies (\sqrt{x_i})_x = 0 \quad \text{in} \quad O_{X,x} \]

is localization of \( X \cup H \) (localization map is injective, because \( X \) is integral).

\( H \cap X \) and \( x \) is a non-singular point of \( X \cup H \).

\[ \implies \quad T_x (X \cap H) \text{ has dim. } d-1. \]

\[ \implies \quad \frac{m_{X \cup H, x}}{m_{X \cup H, x}^2} \text{ has dim. } d-1 \]
Note that:
\[ G_{x \cap H, x} = G_{x, x} / (f / x_i)_x \]

and
\[ m_{x \cap H, x} = m_{x, x} / (f / x_i)_x \]
\[ (x \in x \cap H \iff (f / x_i)_x \in m_{x, x}) \]

and
\[ m_{x \cap H, x} / m_{x \cap H, x} \]
\[ m_{x, x} / m_{x, x} \]
\[ (f / x_i)_x \mod m_{x, x}^2 \]

So, \[ \dim m_{x \cap H, x} / m_{x \cap H, x} = d - 1 \]
\[ (f / x_i)_x \neq 0 \mod m_{x, x}^2 \]
So, if \( x \in X \cup H \) is a singular point, then
\[
\left( \frac{f}{x_i} \right)_x = 0 \quad \text{modulo} \quad \Delta_x.
\]
This proves that \( (x, H) \in B \iff f \in \ker \phi_x \).

**Note:** that if we change the index \( i \), i.e.,
\( x \in U_i \), \( \phi_x \) changes by multiplication by \( \left( \frac{x_i}{x_j} \right)(x) \)
which is a nonzero scalar. So \( \ker \phi_x \) is independent of \( i \).

**Note:** \( \phi_x \) is surjective because \( M_x \) is generated by images \( \langle x, 0, x_i \rangle \) of some degree 1 polynomials, i.e., elements of \( T(C(p_i)(1)) \) (we are over an alg. closed field).

\[
M_x \rightarrow M_x, x, x
\]
If \( x = (a_0, \ldots, a_n) \), then
\[
M_{\mathcal{P}_n, x} = (a_i x_j - a_j x_i)_{0 \leq i, j \leq n}
\]
so \( \ker \varepsilon_x \) has dimension \( n + 1 - (d + 1) = n - d \) over \( k \).

Now:

We show \( q_2(B) \) is a proper closed subset of \( |G_{\mathcal{P}_n}(1)| \).

By Ex. II.3.22 about the fibers of a morphism, the underlying set of \( \pi_1^{-1}(\alpha) \) is the set of preimages of \( \alpha \) by \( \pi_1 \) where
\[
\pi_1^{-1}(\alpha) = \left\{ (x, H) \mid (x, H) \in B \right\}
\]
which equals \( \{x\} \times \text{IP} \ker \varepsilon_x \).
So $\dim q_1^*(x) = \dim (\mathcal{F}_x \times \mathcal{P} \ker \varepsilon_x) = \dim (\mathcal{P} \ker \varepsilon_x) = n - d - 1$

Ex. II.3.22: 

$$\dim B = \dim q_1^*(B) + \dim \tilde{q}_1^*(x) \leq \dim X + n - d - 1 = n - 1$$

Ex. II, 4.4: the image of a proper scheme is proper.

$B$ is a closed subscheme of $X \times \mathcal{P}^n(1)$, hence it is proper (1). So $q_2^*(B)$ is also proper, hence it is closed in $\mathcal{P}^n(1)$, and $\dim q_2^*(B) \leq \dim B \leq n - 1$.

Since $\dim \mathcal{P}^n = n$, $q_2^*(B) \subseteq \mathcal{P}^n(1)$.

\[\Box\]