Theorem: (II.8.15) X separated of finite type over an algebraically closed field k. Then $\Omega^1_{X/k}$ is a locally free sheaf of rank equal to the dimension of X if and only if X is a nonsingular variety $/k$ (assume all irreducible components of X have the same dimension, i.e; X is of pure dimension). The same statement is true if k is not algebraically closed, provided that X is a variety.

One of the main ingredients in the proof is

Lemma (Prop. II.8.7) Let R be a noetherian local ring with maximal ideal m and containing a field k isomorphic to its residue field R/m. $\left(\begin{tikzcd} k \arrow[r, hook] \arrow[rr, bend right, "\cong"'] & R \arrow[r, two heads] & R/m \end{tikzcd} \right)$.

Then the map $\delta : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega^1_{R/k} \otimes_R k$

from the sequence $\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega^1_{R/k} \otimes_R k \longrightarrow \overset{\overset{0}{\shortparallel}}{\Omega^1_{k/k}} \longrightarrow 0$

obtained $k \hookrightarrow R \twoheadrightarrow R/\mathfrak{m} \cong k$

$\left(A \longrightarrow B \longrightarrow C = B/I \right)$

is an isomorphism.

<u>Embedded tangent spaces</u>: (link to "familiar" things)

X a scheme of finite type $/k$ alg. closed.

$x \in X$ a closed point, $U = \operatorname{Spec} A$ an affine open neighborhood of x.

$\Rightarrow \exists \quad A \cong k[x_1, \dots, x_n] / (f_1, \dots, f_r)$

Via this presentation, we view U as the closed subscheme of $\mathbb{A}^n_k$ with ideal $I := (f_1, \ldots, f_r)$

from the tower of rings $k \longrightarrow k[x_1, \ldots, x_n] \twoheadrightarrow A$

we obtain the exact sequence:

$$\begin{array}{ccccccc}
\overbrace{I/I^2} & \xrightarrow{\delta} & \overbrace{\Omega^1_{k[x_1,\ldots,x_n]/k} \otimes A} & \longrightarrow & \overbrace{\Omega^1_{A/k}} & \longrightarrow & 0 \\
\parallel & & \parallel & & \parallel & & \\
\tilde{I}/\tilde{I}^2 & & e^* \Omega^1_{\mathbb{A}^n_k/k} & & \Omega^1_{U/k} & &
\end{array}$$

$$e : U \hookrightarrow \mathbb{A}^n_k$$

After a translation, we can assume $x \leftrightarrow (0, \ldots, 0) \in \mathbb{A}^n$

i.e., $x \leftrightarrow$ maximal ideal $(x_1, \ldots, x_n) = m_x$

From the lemma we have $m_{\mathbb{A}^n, x}/m^2_{\mathbb{A}^n, x} \xrightarrow{\cong} \Omega^1_{\mathbb{A}^n/k, x} \otimes_{\mathcal{O}_{\mathbb{A}^n, x}} k$

We know that $\Omega^1_{\mathbb{A}^n/k} = \mathcal{O}_{\mathbb{A}^n} dx_1 \oplus \cdots \oplus \mathcal{O}_{\mathbb{A}^n} dx_n$

$$\Rightarrow \quad \mathfrak{m}_{\mathbb{A}^n,x} / \mathfrak{m}^2_{\mathbb{A}^n,x} \xrightarrow{\simeq} \left(\mathcal{O}_{\mathbb{A}^n,x}\, dx_1 \oplus \cdots \oplus \mathcal{O}_{\mathbb{A}^n,x}\, dx_n\right) \otimes_{\mathcal{O}_{\mathbb{A}^n,x}} k$$

$$\| $$

$$k\, dx_1 \oplus \cdots \oplus k\, dx_n$$

So we can identify the Zariski cotangent space $\left(T_x \mathbb{A}^n\right)^*$ with the k-vector space with basis $dx_1, \ldots, dx_n$.

$$\left(T_x \mathbb{A}^n\right)^* = \mathfrak{m}_{\mathbb{A}^n,x} / \mathfrak{m}^2_{\mathbb{A}^n,x}$$

Modding out by the equations of V:

$$\Omega^1_{A/k} = \Omega^1_{\mathbb{A}^n/k} / (df_1, \ldots, df_r)$$

$\Rightarrow$ The Zariski cotangent space to V at x can be identified with the quotient of $\underbrace{k\, dx_1 \oplus \cdots \oplus k\, dx_n}_{\Omega^1_{k^n}}$ by the subvector space generated by the images of

$df_1, \dots, df_r$ in $\mathcal{O}_{\mathbb{A}^n, x}/m_x \cong k$

$$\overset{\shortparallel}{k[x_1, \dots, x_n]} \Big/ (x_1, \dots, x_n) \cong k$$

i.e., $df_1(0, \dots, 0), \dots, df_r(0, \dots, 0)$

$$df_i(0, \dots, 0) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(0, \dots, 0)\, dx_j$$

The scheme V (or X) is nonsingular at x iff $\dim_k m_{V,x}/m_{V,x}^2$

$= \dim A =: \dim V =: d$

$\Longleftrightarrow$ the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}(0, \dots, 0)\right)_{\substack{1 \le i \le r \\ 1 \le j \le n}}$

has rank $n - d$.

The projective case: Let $X \subset \mathbb{P}^n_k$ be a quasi-projective scheme $/k$ alg. closed.
By this we mean X is an open subscheme of a closed subscheme Y of $\mathbb{P}^n_k$. Let I_Y be the homogeneous ideal of Y : $I_Y := \bigoplus_{n \geq 0} \Gamma(\mathbb{P}^n_k, \mathcal{I}_Y(n))$

Choose homogeneous generators $P_1, \dots, P_r$ of respective degrees $d_1, \dots, d_r$ for I_Y.

Let $x \in X$ be a closed point. We can assume, after a change of coordinates that x is the origin of $U_0 = D_+(X_0) \cong \mathbb{A}^n_k$

i.e., $\mathfrak{m}_x = \left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0} \right)$ in U_0.

Then $I_{Y \cap U_0} = \left(\frac{P_1}{X_0^{d_1}}, \dots, \frac{P_r}{X_0^{d_r}} \right)$

Since X and Y have the same local rings and the same Zariski cotangent or tangent spaces at x, we replace X with Y and assume X is projective.

We apply the general result we had before:

$\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ can be identified with the quotient of $k\, d\left(\frac{X_1}{X_0}\right) \oplus \cdots \oplus k\, d\left(\frac{X_n}{X_0}\right)$ by the subvector space generated by $d\left(\frac{P_1}{X_0^{d_1}}\right)(0,\dots,0), \dots, d\left(\frac{P_r}{X_0^{d_r}}\right)(0,\dots,0)$

By the Leibnitz rule

$$(*) \quad d\left(\frac{P_i}{X_0^{d_i}}\right) = \frac{X_0\, dP_i - P_i\, dX_0}{X_0^{d_i+1}} \overset{\text{on } X}{=} \frac{dP_i}{X_0^{d_i}}$$

$$dP_i = \sum_{j=0}^{n} \frac{\partial P_i}{\partial X_j}\, dX_j$$

(*) $\Rightarrow$ the rank of $\left(\dfrac{\partial\left(\frac{P_i}{X_0^{d_i}}\right)}{\partial\left(\frac{X_j}{X_0}\right)}(0,\dots,0) \right)_{\substack{1\le i\le r\\ 1\le j\le n}}$

is equal to the rank of $\left(\dfrac{\partial P_i}{\partial X_j}(0,\dots,0) \right)_{\substack{1\le i\le r\\ 0\le j\le n}}$

$\Rightarrow$ the vector space $\dfrac{k\,dX_0 \oplus \cdots \oplus k\,dX_n}{(dP_1(0,\dots,0), \dots, dP_r(0,\dots,0))}$

has dimension 1 more than $\dfrac{k\,d\left(\frac{X_1}{X_0}\right) \oplus \cdots \oplus k\,d\left(\frac{X_n}{X_0}\right)}{\left(d\left(\frac{P_1}{X_0^{d_1}}\right)(0,\dots,0), \dots, d\left(\frac{P_r}{X_0^{d_r}}\right)(0,\dots,0)\right)}$

So X is nonsingular at $x \Leftrightarrow \left(\dfrac{\partial P_i}{\partial X_j}(0,\dots,0)\right)$ has rank $n-d$.

Zariski tangent spaces:

In the general case,

$$k(df_1)_{(x)} + \cdots + k(df_r)_{(x)} \longrightarrow k\,dx_1 \oplus \cdots \oplus k\,dx_n \longrightarrow\!\!\!\!\rightarrow m_{U,x}/m_{U,x}^2 \longrightarrow 0$$

dualize:

$$(k\,df_1(x) + \cdots + k\,df_r(x))^* \longleftarrow T_x\mathbb{A}^n \longleftarrow T_x U \longleftarrow 0$$

$\Rightarrow$ equations for $T_x U \subset T_x\mathbb{A}^n$ are given by (as linear forms on k^n) by $df_1(x), \ldots, df_r(x)$:

$dx_i \;:\; k^n \longrightarrow k$ linear form

$\qquad (a_1, \ldots, a_n) \longmapsto a_i$

$$df_i(x)\,(a_1,\dots,a_n) = \left(\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(x)\, dx_j\right)(a_1,\dots,a_n)$$

$$= \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(x)\; dx_j(a_1,\dots,a_n)$$

$$= \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(x)\; a_j$$

$T_x U \subseteq \overset{\overset{k^n}{\|}}{T_x \mathbb{A}^n}$ is the subspace with equations the above linear forms.

In the projective case, the Zariski tangent space to the affine cone over X at any point y above x is the subspace of $k^{n+1} = T_y \mathbb{A}^{n+1} = (k\,dX_0 \oplus \cdots \oplus k\,dX_n)^*$ with equations $dP_1(y), \dots, dP_m(y)$.