Blow-ups: We blow up a scheme along a coherent sheaf of ideals or along a closed subscheme.

When we do this, we replace the coherent sheaf of ideals with an invertible sheaf, or we replace the closed subscheme with a locally principal Weil divisor (or closed subscheme).

The ambiant scheme becomes "nicer" (examples later).

Def: The blow up $Bl_{\mathcal{I}} X$, resp., $Bl_Y X$ of X along the coherent sheaf of ideals $\mathcal{I}$, resp., the closed subscheme Y, is the scheme $\mathrm{Proj}_X \mathcal{S} \xrightarrow{\pi} X$, where

$$\mathcal{S} := \bigoplus_{d \geq 0} \mathcal{I}^d, \quad \text{resp.,} \quad \mathcal{S} := \bigoplus_{d \geq 0} \mathcal{I}_Y^d$$

and $\mathcal{I}^d$, resp., $\mathcal{I}_Y^d$, is the d-th power of $\mathcal{I}$, resp., $\mathcal{I}_Y$, in $\mathcal{O}_X$.

Note: $\mathcal{I} \subset \mathcal{O}_X$

$$\begin{tikzcd} \mathcal{I}^{\otimes d} \arrow[r] \arrow[d, two heads] & \mathcal{O}_X^{\otimes d} = \mathcal{O}_X \\ \mathcal{I}^{d} \arrow[ur] & \end{tikzcd}$$

Local description of a blow-up:

On any open $U = \operatorname{Spec} A \subset X$, let $I \subset A$ be the global sections of $\mathcal{I}$, i.e., $\tilde{I} = \mathcal{I}|_U$.

Let $a_0, \dots, a_n \in A$ be a generating set for I:

$$\mathcal{S} := \bigoplus_{d \geq 0} \mathcal{I}^d \qquad \mathcal{S}|_U = \bigoplus_{d \geq 0} \tilde{I}^d = \widetilde{\left(\bigoplus_{d \geq 0} I^d\right)}$$

Put $S := \bigoplus_{d \geq 0} I^d$, then $Bl_{\mathcal{I}} X|_U = \operatorname{Proj} S$

We have a surjective homomorphism of A-algebras:

$$\psi : A[X_0, \dots, X_n] \twoheadrightarrow S, \quad X_i \mapsto a_i$$

We have seen that this defines a closed embedding $\varphi : \operatorname{Proj} S \hookrightarrow \operatorname{Proj} A[X_0, \dots, X_n] =: \mathbb{P}^n_A$ and the homogeneous ideal of the image of φ is the kernel of φ, i.e., it is the ideal generated by all the homogeneous polynomials $F \in A[X_0, \dots, X_n]$ s.t. $F(a_0, \dots, a_n) = 0$.

Properties:

(1) Def: the inverse image sheaf of ideals $\mathcal{J}$ on $Bl_{\mathcal{I}} X$ is the image of $\pi^* \mathcal{I} := \pi^{-1}\mathcal{I} \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{O}_{Bl_{\mathcal{I}} X}$ via the natural map

$$\pi^* \mathcal{I} = \pi^{-1}\mathcal{I} \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{O}_{Bl_{\mathcal{I}} X} \longrightarrow \pi^{-1}\mathcal{O}_X \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{O}_{Bl_{\mathcal{I}} X} = \pi^* \mathcal{O}_X = \mathcal{O}_{Bl_{\mathcal{I}} X}$$

On any open $U = \operatorname{Spec} A$, with $I := \Gamma(U, \mathcal{J})$

$$S := \bigoplus_{d \geq 0} I^d$$

$$\widetilde{I} = \mathcal{J}|_U$$

$$\pi^* \mathcal{J}|_U = \widetilde{\left(I \otimes_A S \right)}$$ tilda on Proj (exercise)

$$\mathcal{J}|_U = \widetilde{IS}$$ where IS = image of $I \otimes S$ in S

= ideal generated in S by I

$$IS = I \left(\bigoplus_{d \geq 0} I^d \right) = \bigoplus_{d \geq 0} I^{d+1} \hookrightarrow S[1]$$

= in degrees ≥ 0

$$\Rightarrow \quad \mathcal{J}|_U \cong \widetilde{S[1]} = \mathcal{O}_{\operatorname{Proj} S}(1)$$ invertible sheaf.

can verify that $\mathcal{J}$ = ideal of $\pi^{-1}(Y) \subset Bl_Y X$ when $\mathcal{J} = \mathcal{J}_Y$

$\Rightarrow$ $\pi^{-1}(Y)$ is locally defined by one equation

i.e., $\pi^{-1}(Y)$ is locally principal

<u>Terminology and notation:</u> $\pi^{-1}(Y)$ (= the zero scheme of $\mathcal{J}$)

is the exceptional divisor of the blow up,

often denoted $E := \pi^{-1}(Y)$.

(2) The restriction

$$\begin{array}{ccccc} \pi\big|_{Bl X \setminus E} & : & Bl_Y X \setminus E & \longrightarrow & X \setminus Y \\ & & \cap & & \cap \\ & & Bl_Y X & & X \end{array}$$

is an isomorphism.

Indeed: $\mathcal{D}_Y\big|_{X \setminus Y} = \mathcal{O}_{X\setminus Y} T \Rightarrow \mathcal{S}\big|_{X\setminus Y} = \bigoplus_{d\geq 0} \mathcal{O}^d_{X\setminus Y} T^d$

T a generator

$$= \bigoplus_{d\geq 0} \mathcal{O}_{X\setminus Y} T^d$$

So $\mathcal{S}|_{X\setminus Y} \cong \mathcal{O}_{X\setminus Y}[T]$

$$\Rightarrow \quad \operatorname{Proj} \mathcal{S}|_{X\setminus Y} \cong \operatorname{Proj} \mathcal{O}_{X\setminus Y}[T] = \mathbb{P}^0_{X\setminus Y} = X\setminus Y$$

<u>Conclusion:</u> We turned Y into a Cartier divisor and did not change X away from Y.

<u>Example:</u> We blow up the linear space

$$P := Z(X_0, \dots, X_r) \subset \mathbb{P}^n_k \qquad r < n$$

In $U_i = D_+(X_i) \subset \mathbb{P}^n$, the ideal of P is generated by $\frac{X_0}{X_i}, \dots, \frac{X_r}{X_i}$. We have two possibilities: $i \le r$, or $i > r$

(a) If $i \le r$, then $\frac{X_i}{X_i} = 1$ is one of the generators of I

$\Rightarrow P \cap U_i = \emptyset \qquad \Rightarrow$ the blow up doesn't change U_i.

(b) Suppose $i > r$, say $i = n$.

Then the ideal $I_{P \cap U_n}$ is generated by

$$x_0 := \frac{X_0}{X_n}, \quad x_1 := \frac{X_1}{X_n}, \dots, x_r := \frac{X_r}{X_n}$$

$$(Bl_P X)|_{U_n} = \operatorname{Proj} S \qquad S = \bigoplus_{d \geq 0} I^d_{P \cap U_n}$$

We have the surjective homomorphism of $\mathcal{O}(U_n)$-algebras

$$\mathcal{O}(U_n) = k\left[\frac{X_0}{X_n}, \dots, \frac{X_n}{X_n}\right] = x_0, \dots, 1$$

$$\varphi : \quad k[x_0, \dots, x_{n-1}][Y_0, \dots, Y_r] \twoheadrightarrow S$$

$$Y_i \longmapsto x_i$$

which gives the closed embedding $Bl_P X|_{U_n} \hookrightarrow \mathbb{P}^r_{U_n}$

The homogeneous ideal of $Bl_P X|_{U_n}$ is the kernel of φ, which is generated by $\{x_i Y_j - x_j Y_i \mid 0 \leq i, j \leq r\}$

The simplest case: $n=2$, $r=1$.

$$P = Z(X_0, X_1) \subset \mathbb{P}^2_k \quad \Rightarrow \quad P = \{(0,0,1)\} \subset \mathbb{P}^2$$

In the chart $U_2 = D_+(X_2) \cong \mathbb{A}^2_k$, the point P is the origin $(0,0)$ of $\mathbb{A}^2_k$.

$$U_2 = \operatorname{Spec} k\left[\frac{X_0}{X_2}, \frac{X_1}{X_2}\right] = \operatorname{Spec} k[x_0, x_1]$$

$\mathrm{Bl}_P \mathbb{A}^2 = \mathrm{Bl}_P \mathbb{P}^2 \big|_{U_2}$ is the closed subscheme of

$\mathbb{P}^1_{U_2} \cong \mathbb{P}^1_k \times_{\operatorname{Spec} k} U_2 \cong \mathbb{P}^1_k \times \mathbb{A}^2_k$ defined by the ideal generated by $x_0 Y_1 - x_1 Y_0$.

Write
$$\mathbb{P}^1 \times \mathbb{A}^2 = \underset{\substack{\rotatebox{90}{$\cong$} \\ \mathbb{A}^1 \times \mathbb{A}^2 \cong \mathbb{A}^3}}{D_+(Y_0) \times \mathbb{A}^2} \cup D_+(Y_1) \times \mathbb{A}^2$$

$$D_+(Y_0) \cong \mathbb{A}^1 = \operatorname{Spec} k\left[\frac{Y_1}{Y_0}\right] \quad \text{put } s := \frac{Y_1}{Y_0}$$

$$D_+(Y_0) \times \mathbb{A}^2 = \operatorname{Spec} k[s, x_0, x_1] \cong \mathbb{A}^3$$

$$\mathrm{Bl}_p \mathbb{A}^2 \cap \left(D_+(Y_0) \times \mathbb{A}^2\right) = Z\left(x_0 \frac{Y_1}{Y_0} - x_1\right) = Z(x_0 s - x_1)$$

Similarly $D_+(Y_1) \cong \mathbb{A}_1 = \operatorname{Spec} k\left[\frac{Y_0}{Y_1}\right]$, put $t := \frac{Y_0}{Y_1}$

$$\mathrm{Bl}_p \mathbb{A}^2 \cap \left(D_+(Y_1) \times \mathbb{A}^2\right) = Z\left(x_0 - x_1 \frac{Y_0}{Y_1}\right) = \underset{\mathbb{A}^3}{\underset{\cap}{Z(x_0 - x_1 t)}}$$

<u>glueing</u>: $\operatorname{Spec} k[x_0, x_1, s] \big/ (x_0 s - x_1)$ $\left(= \operatorname{Spec} k[x_0, s] \cong \mathbb{A}^2\right.$

$\updownarrow$ where s, t are invertible

$s \longleftrightarrow t^{-1}$

$x_0 \longmapsto x_1 t$

$x_1 \longmapsto x_0 s$

$\mathbb{A}^2 \cong \operatorname{Spec} k[x_1, t] = \operatorname{Spec} k[x_0, x_1, t] \big/ (x_0 - x_1 t)$

The exceptional divisor $E = \pi^{-1}(P)$

$$\mathcal{I}_E \, (= \mathcal{J}) = \mathcal{O}(1) \text{ generated by the}$$

(inverse) images of the generators of $\mathcal{I}_P$.

So, in the chart $\operatorname{Spec} k[x_0, s]$, the ideal of E is generated by x_0 (x_1 here is a multiple of x_0; $x_1 = x_0 s$)

In the chart, the ideal of E is generated by x_1 ($x_0 = x_1 t$ is a multiple of x_1).