

Prop. 7.2 in Hartshorne.

Theorem: $X/\text{Spec } A$, \mathcal{L} invertible sheaf, $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ generating \mathcal{L} . The associated morphism $\varphi: X \rightarrow \mathbb{P}_A^n$ is a closed embedding iff, $\forall i = 0, \dots, n$, the open set V_i where s_i generates \mathcal{L} is affine and the morphism

$$\varphi_i^\# : A \left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i} \right] \rightarrow \mathcal{O}_X(V_i)$$

is surjective. Note: there is a typo here in Hartshorne

Proof in Hartshorne, a better yet: do it as an exercise.

Theorem: Suppose $A = k$ is an algebraically closed field, X is projective over k (i.e., \exists a closed embedding $X \hookrightarrow \mathbb{P}_k^m$ for some m). Let \mathcal{L} be an invertible sheaf on X , s_0, \dots, s_n global sections of \mathcal{L} generating \mathcal{L} , and let $\varphi: X \rightarrow \mathbb{P}_k^n$

be the associated morphism, $V := \langle s_0, \dots, s_n \rangle \subset T(X, \mathcal{L})$

the k -span of s_0, \dots, s_n .

Then φ is a closed embedding iff

(1) V separates points, i.e., $\forall x, y \in X$ distinct closed points, $\exists s \in V$ s.t. $s(x) = 0$ and $s(y) \neq 0$.

$${}^m \mathcal{L}_x / m_x \mathcal{L}_x \cong k(x) \cong k$$

(2) V separates tangent vectors, i.e., $\forall x \in X$ closed point,

the induced k -linear map

$$\{s \in V \mid s_x \in m_x \mathcal{L}_x (\Leftrightarrow) s(x) = 0\} \longrightarrow m_x \mathcal{L}_x / m_x^2 \mathcal{L}_x \cong m_x / m_x^2$$

is surjective.

as k -vector space

(Proof in Hartshorne)

Remark: Recall from last quarter that the Zariski tangent space to X at x is $(\mathcal{O}_{X,x}/\mathfrak{m}_x^2)^*$ the $k(x)$ -dual of $\mathcal{O}_{X,x}/\mathfrak{m}_x^2 = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}_x^2 = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{X,x}} k(x)$

X projective/ $k \Rightarrow X$ is of finite type/ k

\Rightarrow the residue field of any closed point of X is a finite extension of k

$\Rightarrow \forall x \in X$ closed $k(x) \cong k$ (k alg. closed)

So the Zariski tangent space $T_x X := (\mathcal{O}_{X,x}/\mathfrak{m}_x^2)^*$
 $= \text{Hom}_k(\mathcal{O}_{X,x}/\mathfrak{m}_x^2, k)$
 and $\mathcal{O}_{X,x}^{\otimes 2}/\mathfrak{m}_x^2 \cong \mathcal{O}_{X,x}/\mathfrak{m}_x^2 = T_x^* X$ the Zariski cotangent space.

$s \in V$, $s(x)=0$ means $s_x \in m_x \mathcal{L}_x$

so we can send s to the image of s_x in $m_x \mathcal{L}_x / m_x^2 \mathcal{L}_x$.

This gives the natural map

$$\{s \in V \mid s(x)=0\} \longrightarrow m_x \mathcal{L}_x / m_x^2 \mathcal{L}_x \cong T_x^* X$$

whose surjectivity is equivalent to the injectivity of

the dual map

$$T_x X \longrightarrow \{s \in V \mid s(x)=0\}^* \quad k\text{-dual}$$

Relative Proj:

In exercise II.5.17, you saw the relative Spec of a quasi-coherent sheaf of algebras.

For a quasi-coherent sheaf of algebras \mathcal{A} , we construct $\vec{\text{Spec}}_X \mathcal{A} \rightarrow X$ by gluing the schemes $\text{Spec } \mathcal{A}(U) \rightarrow \text{Spec } \mathcal{O}_X(U)$ for all affine open sets $U \subset X$.

We now construct a relative Proj:

Assume that we have a quasi-coherent sheaf

$\mathcal{S} = \bigoplus_{d \geq 0} \mathcal{S}_d$ of graded \mathcal{O}_X -algebras s.t. $\mathcal{S}_0 = \mathcal{O}_X$,

\mathcal{S}_1 is coherent and \mathcal{S} is locally generated by \mathcal{S}_1 ,

as an \mathcal{O}_X -algebra, i.e., $\forall d > 0$, the multiplication

map $\text{Sym}^d \mathcal{S}_1 \rightarrow \mathcal{S}_d$ is surjective.

Definition: The scheme $\text{Proj}_X \mathcal{I} \rightarrow X$ is constructed by gluing the schemes $\text{Proj} \mathcal{I}(U) \rightarrow U$ for all affine open subschemes $U \subset X$.

On each $\text{Proj} \mathcal{I}(U)$, we have the sheaf $\mathcal{O}_U(1)$.

These glue to give the global twisting sheaf $\mathcal{O}(1)$ on $\text{Proj}_X \mathcal{I}$.

The projective bundle of a locally free sheaf of finite rank:

X noetherian scheme. \mathcal{E} a locally free sheaf of finite rank $n+1$ on X . We saw last quarter the symmetric algebra $\text{Sym} \mathcal{E} := \bigoplus_{u \geq 0} \text{Sym}^u \mathcal{E}$ (homework)

On $U = \text{Spec} A \subset X$ open s.t. $\mathcal{E}|_U \cong \mathcal{O}_X^{\oplus (n+1)}$,

choose a basis X_0, \dots, X_n of $\mathcal{E}|_U$, i.e.,

$$\mathcal{E}|_U = \mathcal{O}_U X_0 \oplus \dots \oplus \mathcal{O}_U X_n$$

then
$$\text{Sym} \mathcal{E}|_U = \bigoplus_{u \geq 0} \text{Sym}^u (\mathcal{O}_U X_0 \oplus \dots \oplus \mathcal{O}_U X_n)$$

$$= \mathcal{O}_U[X_0, \dots, X_n] \text{ the polynomial}$$

algebra on \mathcal{O}_U with generators X_0, \dots, X_n .

Then
$$\text{Spec}_X \text{Sym} \mathcal{E}|_U = \text{Spec} \mathcal{O}_U[X_0, \dots, X_n] = \mathbb{A}^{n+1}_U \rightarrow U$$

and
$$\text{Proj}_X \text{Sym} \mathcal{E}|_U = \text{Proj} \mathcal{O}_U[X_0, \dots, X_n] = \mathbb{P}^n_U \rightarrow U$$

$$\mathcal{O}_X(U)[X_0, \dots, X_n] \longleftarrow \mathcal{O}_X(U)$$

Def: $\mathbb{P}(\mathcal{E}) := \text{Proj}_X \text{Sym } \mathcal{E}$ is the projective bundle associated to \mathcal{E} .

The projections $\text{Proj Sym } \mathcal{E} / U \rightarrow U$ glue to give the natural projection $\mathbb{P}(\mathcal{E}) \xrightarrow{\pi} X$, and we also have the twisting sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

Locally, we saw $\text{Sym}^d \mathcal{E}(U) = (\text{Sym } \mathcal{E}(U))_d = \Gamma(\mathbb{P}_U^n, \mathcal{O}(d))$

$$\Rightarrow \pi_{1*} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d) \cong \text{Sym}^d \mathcal{E}. \quad = \Gamma(U, \pi_{1*} \mathcal{O}(d))$$

$$\Rightarrow \bigoplus_{d \in \mathbb{Z}} \pi_{1*} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d) \cong \text{Sym } \mathcal{E}.$$

Projective morphisms:

Hartshorne defines a projective morphism as a morphism

$f: X \rightarrow Y$ which can be factored into a composition

$$X \xrightarrow{\varphi} \mathbb{P}_Y^n \rightarrow Y \quad \text{for some } n, \text{ where } \varphi$$

is a closed embedding.

Grothendieck defines a projective morphism as a morphism which can be factored into $X \xrightarrow{\varphi} P \rightarrow Y$

where P is a projective bundle over Y and φ is a closed embedding.

Def: A projective n -space bundle over Y is a scheme

$\pi: P \rightarrow Y$ s.t. \exists covering $Y = \bigcup_{i \in I} U_i$ by

open sets $U_i = \text{Spec } A_i$ s.t. $\forall i \exists$

$$\varphi_i : \pi^{-1}(U_i) \xrightarrow{\cong} \mathbb{P}^n_{U_i}$$

and $\forall i, j$ and all open affine $V = \text{Spec } A \subset U_i \cap U_j$,

the isomorphism $(\varphi_j \circ \varphi_i^{-1})^\# : A[x_0, \dots, x_n] \xrightarrow{\cong} A[x_0, \dots, x_n]$

is A -linear.

Note (Ex. #5.18) Every vector bundle is the Spec of the symmetric algebra of some locally free sheaf. However, not every projective bundle is the Proj of the symmetric algebra of some locally free sheaf.

This is the case for "nice" schemes, e.g., regular schemes.

The Hartshorne and Grothendieck definitions coincide for schemes Y that are quasi-projective over an affine scheme.

" "
open subschemes of
projective schemes.

Another nice application of relative Proj: Blow ups.