

Claim: The morphisms φ_i glue together to give a morphism $\varphi: X \rightarrow \mathbb{P}^n$.

We need to verify that $\varphi_i|_{V_i \cap V_j} = \varphi_j|_{V_i \cap V_j}$.

$$\begin{aligned}
 V_i \cap V_j &\xrightarrow[\varphi_j]{\varphi_i} V_i \cap V_j = \text{Spec } A\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right]\left[\left(\frac{X_j}{X_i}\right)^{-1}\right] \\
 &= \text{Spec } A\left[\frac{X_0}{X_j}, \dots, \frac{X_n}{X_j}\right]\left[\left(\frac{X_i}{X_j}\right)^{-1}\right]
 \end{aligned}$$

The identification between the two rings:

$$\begin{aligned}
 A\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right]\left[\left(\frac{X_j}{X_i}\right)^{-1}\right] &\xrightarrow{\cong} A\left[\frac{X_0}{X_j}, \dots, \frac{X_n}{X_j}\right]\left[\left(\frac{X_i}{X_j}\right)^{-1}\right] \\
 \forall l \quad \frac{X_l}{X_i} &\longleftarrow \frac{X_l}{X_j} \left(\frac{X_i}{X_j}\right)^{-1} \\
 \frac{X_l}{X_i} \cdot \left(\frac{X_j}{X_i}\right)^{-1} &\longleftarrow \frac{X_l}{X_j}
 \end{aligned}$$

$$\varphi_i^\# / \nu_i \nu_j : \frac{X_l}{X_i} \mapsto t_{li} \quad \left(\frac{X_j}{X_i}\right)^{-1} \mapsto (t_{ji})^{-1}$$

$$\varphi_j^\# / \nu_i \nu_j : \frac{X_l}{X_j} \mapsto t_{lj} \quad \left(\frac{X_i}{X_j}\right)^{-1} \mapsto (t_{ij})^{-1}$$

We need to verify that $\varphi_i^\# \left(\frac{X_l}{X_i} \right) = \varphi_j^\# \left(\frac{X_l}{X_j} \left(\frac{X_i}{X_j} \right)^{-1} \right)$

(and similarly after switching i and j)

$$\varphi_i^\# \left(\frac{X_l}{X_i} \right) = t_{li} \quad \varphi_j^\# \left(\frac{X_l}{X_j} \left(\frac{X_i}{X_j} \right)^{-1} \right) = t_{lj} \cdot (t_{ij})^{-1}$$

Claim: $(t_{ji})^{-1} = t_{ij}$ and $\forall i, j, l \quad t_{ij} \cdot t_{jl} = t_{il}$

Note that the claim will imply $\varphi_i^\# / \nu_i \nu_j = \varphi_j^\# / \nu_i \nu_j$

Proof of the claim: On $V_i \cap V_j$:

$$\mathcal{L}|_{V_i \cap V_j} = \mathcal{O}_{V_i \cap V_j} \cdot s_i|_{V_i \cap V_j} = \mathcal{O}_{V_i \cap V_j} \cdot s_j|_{V_i \cap V_j}$$

and $s_i|_{V_i \cap V_j} = t_{ij}|_{V_i \cap V_j} \cdot s_j|_{V_i \cap V_j} = t_{ij}|_{V_i \cap V_j} \cdot t_{ji}|_{V_i \cap V_j} \cdot s_i|_{V_i \cap V_j}$

s_i is a generator of a free rank 1 module

$$\Rightarrow t_{ij}|_{V_i \cap V_j} \cdot t_{ji}|_{V_i \cap V_j} = 1$$

$$\begin{aligned} s_l|_{V_i \cap V_j} &= t_{li}|_{V_i \cap V_j} \cdot s_i|_{V_i \cap V_j} = t_{lj}|_{V_i \cap V_j} \cdot s_j|_{V_i \cap V_j} \\ &= t_{lj}|_{V_i \cap V_j} \cdot t_{ji}|_{V_i \cap V_j} \cdot s_i|_{V_i \cap V_j} \end{aligned}$$

$$\Rightarrow t_{li} / v_i v_j = t_{lj}^{-1} / v_i v_j = t_{ji} / v_i v_j \quad \checkmark$$

So the φ_i glue to give $\varphi: X \rightarrow \mathbb{P}^n$

Note that on U_i : $\mathcal{O}_{\mathbb{P}^n}(1) / U_i = \mathcal{O}_{U_i} X_i$

$$\text{when we pull back } \varphi_i^* (\mathcal{O}_{\mathbb{P}^n}(1) / U_i) = \varphi_i^* \mathcal{O}_{U_i} X_i \\ = \mathcal{O}_{V_i} \varphi_i^* X_i$$

$$\text{So, on } V_i \quad \varphi^* \mathcal{O}_{\mathbb{P}^n}(1) / V_i = \varphi_i^* (\mathcal{O}_{\mathbb{P}^n}(1) / U_i)$$

is trivial.

The transition functions of $\varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$ are $\varphi^* \left(\frac{X_i}{X_j} \right)$
 we defined $\varphi^* \left(\frac{X_i}{X_j} \right) = \varphi^\# \left(\frac{X_i}{X_j} \right) = \varphi^\# \left(\frac{X_i}{X_j} \right) = t_{ij}$

The f_{ij} are the transition functions of \mathcal{L} .

So, if we define isomorphisms:

$$\mathcal{L}|_{V_i} = \mathcal{O}_{V_i}(s_i|_{V_i}) \implies \varphi^*(\mathcal{O}_{U_i}(X_i)) = \varphi^*(\mathcal{O}_{\mathbb{P}^n}(1)|_{U_i})$$
$$s_i|_{V_i} \longleftrightarrow \varphi^* X_i|_{U_i}$$

these isomorphisms glue to a global isomorphism

$$\mathcal{L} \cong \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$$

sending $s_i \longleftrightarrow \varphi^* X_i$

$$(2) \quad \varphi: X \longrightarrow \mathbb{P}^n, \text{ put } \mathcal{L} := \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$$
$$s_i := \varphi^* X_i, \quad V_i := \varphi^{-1}(U_i)$$

On U_j , we have $\frac{X_i}{X_j} \in \mathcal{O}_{\mathbb{P}^n}(U_j) = A\left[\frac{X_0}{X_j}, \dots, \frac{X_n}{X_j}\right]$

put $t_{ij} := \varphi^*\left(\frac{X_i}{X_j}\right)$

we have $X_i|_{U_j} = \left(\frac{X_i}{X_j}\right) X_j|_{U_j}$

\Rightarrow
(pull back)

$$s_i|_{U_j} = t_{ij} s_j|_{U_j}$$

and $\mathcal{L}|_{U_j} = (\varphi^* \mathcal{O}_{\mathbb{P}^n}(1))|_{U_j} = \varphi^*(\mathcal{O}_{\mathbb{P}^n}(1)|_{U_j})$

$$= \varphi^*(\mathcal{O}_{U_j} X_j|_{U_j}) = \mathcal{O}_{U_j} \varphi^* X_j|_{U_j}$$

$$= \mathcal{O}_{U_j} s_j|_{U_j}$$

$\Rightarrow s_j$ generates \mathcal{L} on V_j .

Since $\mathbb{P}^n = \bigcup_{i=0}^n U_i \Rightarrow X = \varphi^{-1}(\pi^{-1}) = \bigcup_{i=0}^n V_i$

$\Rightarrow s_0, \dots, s_n$ generate \mathcal{L} on X

Since $\varphi^* \left(\frac{X_i}{X_j} \right) = t_{ij}$, φ is the morphism from part 1.

(determines the morphism on each U_i)

□

Remark: In practice, collections of global sections often do not generate \mathcal{L} . However, we can still consider the open set $U \subset X$ where they do generate. So we obtain a morphism $U \rightarrow \mathbb{P}^n$.

Example: Projections from one projective space to another.

A a ring $S := A[X_0, \dots, X_n] = \bigoplus_{d \geq 0} S_d$

Choose $L_0, \dots, L_n \in S_1$

$P := Z(L_0) \cap \dots \cap Z(L_n)$ is, by definition, a linear subspace of \mathbb{P}^n .

Recall: $I_{Z(L_i)} = \langle L_i \rangle \subset S$

$\Rightarrow I_P = \langle L_0, \dots, L_n \rangle \subset S$

Assume now that A is a field k . $S_1 = kX_0 \oplus \dots \oplus kX_n$

After possibly removing some of the L_i , we can assume they are linearly independent k .

Then (see Chapter 1. Ex. 2.11) P has codim. $n+1$

n dim. $n-1-n$.

After a linear change of coordinates, we can assume

$$L_0 = X_0, \dots, L_n = X_n.$$

$$\text{Then } P = Z(X_0, \dots, X_n) \cong \mathbb{P}_k^{n-1-n} \subset \mathbb{P}_k^n$$

We project \mathbb{P}^n from P (P is the center of the projection)

$$\text{Choose } X = \mathbb{P}^n, \quad \mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)$$

$$s_0 = L_0 = X_0, \dots, s_n = L_n = X_n$$

The open subset of \mathbb{P}^n where s_0, \dots, s_n generate $\mathcal{O}_{\mathbb{P}^n}(1)$ is

$$U := \mathbb{P}^n \setminus P = U_0 \cup \dots \cup U_n$$

By the theorem, we have a morphism

$$\varphi: U \longrightarrow \mathbb{P}_k^n \quad \text{s.t.} \quad \varphi^* \mathcal{O}_{\mathbb{P}^n}(1) \cong \mathcal{O}_U(1)$$

We often write $\varphi: X \dashrightarrow \mathbb{P}^n$ to indicate that

we have a morphism well defined on some open subset of X . We call this a rational map.

Application to quadrics: $A = k$ a field of char. $\neq 2$.

Choose $F \in S_2$

$Q := Z(F) \subset \mathbb{P}^n$ is, by definition, a quadric.

We can use the Gram-Schmidt process to write

$$F = \sum_{i=0}^n L_i^2 \quad \text{where } L_0, \dots, L_n \text{ are lin independent linear polynomials.}$$

The integer $n+1$ is the rank of F or $Q := Z(F)$.

The linear space $P = Z(L_0, \dots, L_n)$ is the vertex of Q .

Via the projection of center P , the quadric Q is the inverse image of a quadric $\bar{Q} \subset \mathbb{P}^{\hat{n}}$:

$$\begin{array}{ccc} \mathbb{P}^{\hat{n}} & \xrightarrow{\varphi} & \mathbb{P}^{\hat{n}} \\ \cup & & \cup \\ Q & \xrightarrow{\varphi|_Q} & \bar{Q} \end{array}$$

$$\text{i.e., } \varphi^{-1}(\bar{Q}) = Q \setminus P \quad \varphi: \mathbb{P}^{\hat{n}} \setminus P \rightarrow \mathbb{P}^{\hat{n}}$$

Let $Y_0, \dots, Y_{\hat{n}}$ be the coordinates on $\mathbb{P}^{\hat{n}}$: $\varphi^* Y_i = L_i$,

$$\text{define } \bar{Q} := Z\left(\sum_{i=0}^{\hat{n}} Y_i^2\right) \quad \varphi^* \bar{Q} = Z\left(\varphi^* \sum_{i=0}^{\hat{n}} Y_i^2\right) = Q \setminus P$$

\bar{Q} has maximal rank (in \mathbb{P}^n), i.e., its vertex is \emptyset .
We will see that P is the singular locus of Q .

Applying this to \bar{Q} : \bar{Q} is nonsingular, i.e., all the local rings of \bar{Q} are regular local rings.

Ampleness and very ampleness:

Def: (1) An invertible sheaf \mathcal{L} on a scheme $X/\text{Spec } A$ is called very ample if there exists a collection $\{s_0, \dots, s_n\}$ of global sections of \mathcal{L} for which the associated morphism is an embedding.

(2) An invertible sheaf is called ample if, for every

coherent sheaf \mathcal{F} on X , $\exists n_0 > 0$ s.t.

$\forall n \geq n_0$ $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections.

Theorem: Suppose X is of finite type / $\text{Spec } A$ \mathcal{L} invertible
Then \mathcal{L} is ample iff $\exists n > 0$ s.t. \mathcal{L}^n is very ample.

(proof in Hartshorne)