We have that $W \cap U = \sum_{i=1}^{m} n_i [Y_i \cap U]$ is the divisor of $f_U := \prod_{i=1}^{m} a_i^{n_i}$, i.e., $W \cap U = D(f_U)$.

Because: the image of a_i in the local ring of the generic point η_i of Y_i generates the maximal ideal of $\mathcal{O}_{X,\eta_i}$ because $(a_i) = \mathfrak{p}_i$ $\mathfrak{p}_i A_{\mathfrak{p}_i} = m_{X,\eta_i}$.

We have a well-defined Cartier divisor $F(W)$ represented by $\{(U, f_U)\}$, because, for two affine open sets U, V as above, f_U / f_V is invertible: $U \cap V$ is affine because X is separated, f_U and f_V generate the same ideal in the ring of $U \cap V$.

By construction the maps D and F are inverses to each other and principal divisors go to principal divisors (think about this a little bit at home). ☐

<u>Remark</u>: The morphism D is well-defined without assuming X locally factorial.
The proof above shows that we can think of Cartier divisors as locally principal Weil divisors (i.e., every point has a neighborhood on which the Weil divisor is principal).
What we proved is that when X is locally factorial, all Weil divisors are locally principal.
This is "the same" as saying that $Cl(\operatorname{Spec} A) = 0$ when A is a UFD.

Example: $\mathbb{P}^n_k$ is locally factorial.

$$\Rightarrow \quad Cl(\mathbb{P}^n_k) \cong CaCl(\mathbb{P}^n_k) \cong Pic(\mathbb{P}^n_k)$$

$$Cl(\mathbb{P}^n_k) = \mathbb{Z}[Z_0] = \mathbb{Z}[Z(X_0)] = \mathbb{Z}[\mathcal{O}_{\mathbb{P}^n}(1)] = Pic(\mathbb{P}^n_k)$$

Effective divisors:

Def: A Cartier divisor is effective if it can be represented by $\{(f_i, U_i)\}$ s.t. $\forall i$ $f_i \in \mathcal{O}(U_i) \subset \mathcal{K}_X(U_i)$.

A Weil divisor $C = \sum_{i=1}^{m} n_i [Y_i]$ is effective if $\forall i, n_i \geq 0$.

The proof of the previous proposition shows that on a noetherian, integral, separated, locally factorial scheme, effective Cartier divisors correspond to effective Weil divisors.

Given an effective Cartier divisor C def. by $\{(U_i, f_i)\}$, we define the associated locally principal subscheme $Z(C)$ to be the subscheme whose sheaf of ideals is generated by f_i on U_i: $\mathcal{I}_{Z(C) \cap U_i} := \mathcal{O}_{U_i} f_i \subset \mathcal{O}_{U_i}$.

In the proof of the proposition, we had $f_U = \prod_{i=1}^{m} a_i^{n_i}$ and

$$I_{Z(C) \cap U} = \langle f_U \rangle \subset A.$$

These glue together to define $\mathcal{I}_{Z(C)} \subset \mathcal{O}_X$ because on any $U_i \cap U_j$: f_i / f_j is invertible $\Longrightarrow$

$$\mathcal{O}_{U_i \cap U_j} f_i = \mathcal{O}_{U_i \cap U_j} f_j \subset \mathcal{O}_{U_i \cap U_j}.$$

Recall that we defined $\mathcal{O}_X(C)$ as the sub-sheaf of $\mathcal{K}_X$ generated on U_i by f_i^{-1}.

This means $\mathcal{I}_{Z(C)} = \mathcal{O}_X(-C)$ by def. when C is effective. So $\mathcal{O}_X(-C) \subset \mathcal{O}_X \subset \mathcal{O}_X(C) \subset \mathcal{K}_X$

(locally $\mathcal{O}_{U_i} \subset \mathcal{O}_{U_i} \cdot f_i^{-1} \subset \mathcal{K}_{U_i}$)

Morphisms to projective space: (everything noetherian)

We fix a ring A, $\mathbb{P}^n := \mathbb{P}^n_A$

Theorem: Suppose X is a scheme over A ($X \to \operatorname{Spec} A$)

(1) Given an invertible sheaf $\mathcal{L}$ on X and global sections $s_0, \dots, s_n$ of $\mathcal{L}$ which generate $\mathcal{L}$, there exists a unique A-morphism $\varphi : X \to \mathbb{P}^n_A$ s.t. $\mathcal{L} \cong \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$ and $\forall i$ $s_i = \varphi^* x_i$.

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}^n_A \\ & \searrow \;\circlearrowleft\; \swarrow & \\ & \operatorname{Spec} A & \end{array}$$

(2) Given an A-morphism $\varphi : X \longrightarrow \mathbb{P}^n$, put $\mathcal{L} := \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$ and $s_i := \varphi^* X_i$ $\forall i = 0, \dots, n$. Then the sections s_i generate $\mathcal{L}$ and φ is the morphism from (1) associated to $\mathcal{L}$ and $s_0, \dots, s_n$.

<u>Proof</u>: (1) For each i, let

$$V_i := \{ x \in X \mid s_i(x) \notin m_x \mathcal{L}_x \}$$

be the open set of X where s_i generates $\mathcal{L}$ (i.e., $s_i(x)$ generates $\mathcal{L}_x$) : $\forall x \in V_i$ $\mathcal{L}_x = \mathcal{O}_{X,x} s_i(x)$. $(m_x \subset \mathcal{O}_{X,x})$

$$\Rightarrow \mathcal{L}|_{V_i} \xleftarrow{\cong} \mathcal{O}_{V_i} \, s_i|_{V_i}$$

Fix i, $\forall j$ $\exists$ $t_{ji} \in \mathcal{O}_X(V_i)$ s.t. $s_j|_{V_i} = t_{ji} \, s_i|_{V_i}$

Define $\varphi_i : V_i \longrightarrow U_i = \operatorname{Spec} A\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right] \subset \mathbb{P}^n$

by the morphism of global sections (of A-alg.)

$$\varphi_i^{\#} : A\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right] \longrightarrow \mathcal{O}_X(V_i)$$

$$\frac{X_j}{X_i} \longmapsto t_{ji} \in \mathcal{O}_X(V_i)$$