We have that \( W \cap U = \sum_{i=1}^{m} m_i [X_i \cap U] \)

is the divisor of \( f_U := \prod_{i=1}^{m} \alpha_i \), i.e., \( W \cap U = D(f_U) \).

Because the image of \( \alpha_i \) in the local ring of the generic point of \( X_i \) generates the maximal ideal of \( \mathcal{O}_{X_i, \eta_i} \),

because \( (\alpha_i) = \mathfrak{p}_i \), \( \mathfrak{p}_i \cap \mathcal{A}_{\mathcal{X}, \eta_i} = \mathfrak{m}_{X, \eta_i} \).

We have a well-defined Cartier divisor represented by \( \{(U, f_U)\} \), because, for two affine open sets \( U, V \)
as above, \( f_U / f_V \) is invertible. \( U \cap V \) is affine because \( X \) is separated, \( f_U \) and \( f_V \) generate the same ideal in the ring of \( U \cap V \).
By construction the maps $D$ and $F$ are inverses to each other and principal divisors go to principal divisors (think about this a little bit at home).

Remark: The morphism $D$ is well-defined without assuming $X$ locally factorial.

The proof above shows that we can think of Cartier divisors as locally principal Weil divisors (i.e., every point has a neighborhood on which the Weil divisor is principal).

What we proved is that when $X$ is locally factorial, all Weil divisors are locally principal.

This is “the same” as saying that $\text{Cl}(\mathcal{O}_{\text{Spec} A}) = 0$ when $A$ is a UFD.
Example: $\mathbf{P}^n_k$ is locally factorial.

$$\Rightarrow \mathcal{O}(\mathbf{P}^n_k) \cong \text{CaCl}(\mathbf{P}^n_k) \cong \text{Pic}(\mathbf{P}^n_k)$$

$$\mathbb{Z}[\mathbb{Z}_0] = \mathbb{Z}[\mathbb{Z}(x_0)] = \mathbb{Z}[\mathcal{O}_{\mathbf{P}^n_k}(1)]$$

**Effective divisors:**

**Def.** A Cartier divisor is effective if it can be represented by $\{(f_i, U_i) \mid \text{ s.t. } \forall i \ f_i \in \mathcal{O}(U_i) \subset \mathbb{K}_x(U_i)$.

A Weil divisor $C = \sum_{i=1}^m n_i [X_i]$ is effective if $\forall i, n_i \geq 0$.

The proof of the previous proposition shows that in a noetherian, integral, separated, locally factorial scheme, effective Cartier divisors correspond to effective Weil divisors.
Given an effective Cartier divisor $C$ resp. by $\{ (U_i, f_i) \}$, we define the associated locally principal subsheave $\mathcal{Z}(C)$ to be the subsheave whose sheaf of ideals is generated by $f_i$ on $U_i$: $\mathcal{I}_{\mathcal{Z}(C) \cap U_i} := \mathcal{O}_{U_i} f_i \subset \mathcal{O}_{U_i}$.

In the proof of the proposition, we had $f = \prod_{i=1}^{n} a_i^{m_i}$ and

$$\mathcal{I}_{\mathcal{Z}(C) \cap U} = \langle f \rangle \subset A.$$

These glue together to define $\mathcal{I}_{\mathcal{Z}(C)} \subset \mathcal{O}_X$ because in any $U_i \cap U_j$, $f_i/f_j$ is invertible $\Rightarrow$

$$\mathcal{O}_{U_i \cap U_j} \cdot f_i = \mathcal{O}_{U_i \cap U_j} \cdot f_j \subset \mathcal{O}_{U_i \cap U_j}.$$

Recall that we defined $\mathcal{O}_X(\mathcal{C})$ as the subsheaf of $\mathcal{K}_X$ generated in $U_i$ by $f_i$. 
This means $\mathbb{Z}(C) = G \times (-C)$ by def.

when $C$ is effective. So $G \times (-C) \subseteq G \times C \subseteq G \times (C) \subseteq \mathbb{P}^n$

(locally $G_{U_i} \subseteq G_{U_i} \subseteq \mathbb{P}^n_{U_i}$)

Morphisms to projective space: (everything noetherian)

We fix a ring $A$, $\mathbb{P}^n = \mathbb{P}^n_A$

Theorem: Suppose $X$ is a scheme over $A$ ($X \to \text{Spec} A$)

(1) Given an invertible sheaf $\mathcal{L}$ on $X$ and global sections $s_0, \ldots, s_n$ of $\mathcal{L}$ which generate $\mathcal{L}$, there exists a unique $A$-morphism $\varphi : X \to \mathbb{P}^n_A$ s.t. $\mathcal{L} \cong \varphi^* O_{\mathbb{P}^n_A}$

and $\forall i \quad s_i = \varphi^* X_i$. 

$\xymatrix{ \mathbb{P}^n_A \ar[r]^\varphi \ar[d] & \text{Spec} A \ar[d] \cr X \ar[r] & X }$
(2) Given a morphism \( \varphi : X \to \mathbb{P}^n \), put
\[ L := \varphi^* \mathcal{O}_{\mathbb{P}^n}(1) \] and \( s_i := \varphi^* X_i \quad \forall i = 0, \ldots, n \). Then the sections \( s_i \) generate \( L \) and \( \varphi \) is the morphism from (1) associated to \( L \) and \( s_0, \ldots, s_n \).

**Proof:** (1) For each \( i \), let
\[ V_i := \{ x \in X \mid s_i(x) \notin m_x \mathcal{L}_x \} \]
be the open set of \( X \) where \( s_i \) generates \( \mathcal{L} \) (i.e., \( s_i(x) \) generates \( \mathcal{L}_x \)):
\[ \forall x \in V_i : \mathcal{L}_x = \mathcal{G}_{X, x} s_i(x). \quad (m_x \subset \mathcal{G}_{X, x}) \]

\[ \Rightarrow \mathcal{L} \mid V_i \cong \mathcal{G}_{V_i} s_i \mid V_i \]

Fix \( i \), \( \forall j : \exists \, t_{ji} \in \mathcal{O}_X (V_i) \) s.t. \( s_j \mid V_i = t_{ji} s_i \mid V_i \).
Define $\varphi_i : V_i \to U_i = \text{Spec} A[\frac{X_0}{X_i}, \ldots, \frac{X_n}{X_i}] \subset \mathbb{P}^n$ by the morphism of global sections (of $A_{-\text{alg}}$)

$$
\varphi_i : A[\frac{X_0}{X_i}, \ldots, \frac{X_n}{X_i}] \to \mathcal{O}_X(V_i)
$$

$$
\frac{X_j}{X_i} \longmapsto t_{ji}
$$