By construction, $\mathcal{O}_X(D) \cong \mathcal{O}_X \iff D$ is principal

<u>Also note</u>: multiplication of Cartier divisors

$\longleftrightarrow$ tensor product of invertible sheaves

(<u>locally</u>: $\mathcal{O}_{U_i} f_i^{-1} \otimes_{\mathcal{O}_{U_i}} \mathcal{O}_{U_i} g_i^{-1} \cong \mathcal{O}_{U_i} f_i^{-1} g_i^{-1}$
$=$ as subsheaves of $\mathcal{K}_{U_i}$)

$-D$ represented by $\{(f_i^{-1}, U_i)\}$ if D is rep. by $\{(f_i, U_i)\}$

$\Rightarrow \mathcal{O}_X(-D)$ is locally $\mathcal{O}_{U_i} f_i$

$\Rightarrow \quad \mathcal{O}_X(D) \cong \mathcal{O}_X(D') \iff \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')^{-1} \cong \mathcal{O}_X$

$\iff \mathcal{O}_X(D) \otimes \mathcal{O}_X(-D') \cong \mathcal{O}_X \iff \mathcal{O}_X(D-D') \cong \mathcal{O}_X$

$\iff D-D'$ is principal $\iff D$ is linearly eq. to D'.

We have $\mathrm{Pic}(X) :=$ group of invertible sheaves on X modulo isomorphism.

$\mathrm{Pic}^{rat}(X) :=$ group of invertible subsheaves of $\mathcal{K}_X$ modulo isomorphism

$\mathrm{CaCl}(X) :=$ group of Cartier divisors on X modulo linear equivalence.

We just saw that $\mathrm{CaCl}(X) \cong \mathrm{Pic}^{rat}(X)$
$(\hookrightarrow \mathrm{Pic}(X))$

<u>Lemma:</u> If X is integral, then
$$\mathrm{Pic}^{rat}(X) = \mathrm{Pic}(X).$$

<u>Proof:</u> We need to show that every invertible sheaf is isomorphic to an invertible subsheaf of $\mathcal{K}_X$.

Let $\mathcal{L}$ be an invertible sheaf.

$$\mathcal{O}_X \hookrightarrow \mathcal{K}_X$$

$$\Rightarrow \quad \begin{array}{c} \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{L} \\ \wr\Vert \\ \mathcal{L} \end{array} \longrightarrow \mathcal{K}_X \otimes \mathcal{L}$$

<u>Claim</u> $\mathcal{K}_X \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{K}_X$, because

X is integral $\Rightarrow$ $\mathcal{K}_X$ is the constant sheaf with group $K(X) = \mathcal{O}_{X,\eta}$ η = generic point

we verify that $\mathcal{K}_X \otimes_{\mathcal{O}_X} \mathcal{L}$ is also the constant sheaf with group $K(X)$.

On any open set U where $\mathcal{L}$ is trivial, we have $(\mathcal{K}_X \otimes \mathcal{L})|_U \cong \mathcal{K}_X|_U \otimes \mathcal{L}|_U \cong \mathcal{K}_X|_U \cong \mathcal{K}_U$

$\mathcal{K}_U$ is the constant sheaf on U with group $K(U) = K(X)$.

If $V \subset X$ is any open set, then $V = \bigcup_{i \in I} V \cap U_i$

where U_i is affine and $\mathcal{L}|_{U_i} \simeq \mathcal{O}_{U_i}$.

Then if $s \in \Gamma(V, \mathcal{K}_X \otimes \mathcal{L})$

$$s \Longleftrightarrow \{ s_i := s|_{V \cap U_i} \}$$

$$s_i|_{V \cap U_i \cap U_j} = s_j|_{V \cap U_i \cap U_j} = s|_{V \cap U_i \cap U_j} \in K(X)$$

$$\| \qquad K(V \cap U_i \cap U_j).$$

$\Rightarrow s \in K(X)$

$\Rightarrow \quad \mathcal{K}_X \otimes \mathcal{L}\,(V) = K(X)$. $\square$

Cartier versus Weil divisors.

<u>Proposition</u>: Suppose X is integral, separated, locally factorial (i.e., all local rings of X are UFD). Then, the additive group $\mathrm{Div}(X)$ is isomorphic to the multiplicative group $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$.

Under this isomorphism, principal divisors map to principal divisors, i.e., $\mathrm{Cl}(X) \cong \mathrm{CaCl}(X)$ $(\cong \mathrm{Pic}(X))$

<u>Proof</u>: We construct morphisms of groups

$$D: \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \longrightarrow \mathrm{Div}(X)$$

$$F: \mathrm{Div}(X) \longrightarrow \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$$

which are inverses of each other and send principal divisors to principal divisors.

Definition of D: X is integral, so $\mathcal{K}_X$ is the constant sheaf with group $K(X)$. Let C be a Cartier divisor represented by $\{(f_i, U_i), X = \bigcup_{i \in I} U_i, f_i \in K(X)\}$

$$D(C) := \sum_{\substack{Y \subset X \\ \text{integral divisor, } i \text{ s.t. } Y \cap U_i \neq \emptyset}} v_Y(f_i)\,[Y]$$

this is well-defined because if $Y \cap U_i \cap U_j \neq \emptyset$, then

$$f_i/f_j \in \mathcal{O}^*(U_i \cap U_j) \Rightarrow v_Y(f_i/f_j) = 0$$
$$\Rightarrow v_Y(f_i) = v_Y(f_j).$$

the sum is finite because X is noetherian: we can cover X with a finite number of the U_i and on each U_i, only a finite # of $v_Y(f_i)$ are nonzero.

Definition of F: Let $W = \sum_{i=1}^{m} n_i Y_i$ be a Weil divisor. Let $x \in X$ be any point. Choose an affine open neighborhood of x, say $U = \operatorname{Spec} A$ s.t. if $x \notin Y_i$, then $U \cap Y_i = \phi$. $x \longleftrightarrow \mathfrak{q} \subset A$

$\forall i$ s.t. $x \in Y_i$ $Y_i \cap U \neq \phi$

and $Y_i \cap U = V(\mathfrak{p}_i)$ where $\mathfrak{p}_i \subset A$ is prime.

$Y_i \cap U \subset U$ has codim. 1 means $\mathfrak{p}_i$ has height 1.

Choose a set of generators $a_{1i}, \dots, a_{ni}$ of $\mathfrak{p}_i$.

$\forall i, j$ the germ $a_{ij}(x) \in \mathcal{O}_{X,x}$ is a product of its irreducible factors, one of which belongs to the ideal $(\mathfrak{p}_i)_x$ generated by the image of $\mathfrak{p}_i$ in $\mathcal{O}_{X,x} = A_{\mathfrak{q}}$

Shrinking U if necessary (localize if necessary), we can assume $a_{ij^\cdot}(x)$ is irreducible.

$a_{ij^\cdot}(x) = \frac{a_{ij^\cdot}}{1} \in A_{\mathfrak{q}}$ $\quad a_{ij^\cdot}(x) = \prod_r \mu_r \in A_{\mathfrak{q}}$.

$\exists\, r \quad \mu_r \in (\mathfrak{p}_i)_x \subset \mathcal{O}_{X,x} = A_{\mathfrak{q}}$

$\underline{\text{choose one } \mu_r}$

every other $\mu_s = \frac{g}{h} \in A_{\mathfrak{q}}$. $\quad g, h \in A$

$s \neq r$

invert all these $g, h \rightsquigarrow A[g^{-1}h^{-1}\dots]$

replace U with $\operatorname{Spec} A[g^{-1}h^{-1}\dots]$,

Now $\mathcal{O}_{X,x}(a_{ij^\cdot})_x$ is a prime ideal $\subset (\mathfrak{p}_i)_x$

since $(\mathfrak{p}_i)_x$ has height 1 $\Rightarrow$ $\mathcal{O}_{X,x}(a_{ij^\cdot})_x = (\mathfrak{p}_i)_x$

$\Rightarrow \forall\, j, k \qquad (a_{ij})_x$ and $(a_{ik})_x$ are proportional via a unit.

pick one of the a_{ij} and call it a_i.

Shrinking U if necessary, we can assume that

$$A a_i = \mathfrak{p}_i :$$

We know $A(a_i)_x = (\mathfrak{p}_i)_x$ and $Aa_{i1} + \dots Aa_{in} = \mathfrak{p}_i$

and $\forall\ k$ $(a_{ik})_x = \text{unit} \cdot (a_i)_x$. the unit $\in A_{\mathfrak{q}}$

$= \frac{g}{h}$, $g, h \in A$

$$(a_{ik})_x = \frac{a_{ik}}{1} \in A_{\mathfrak{q}}.$$

$$\frac{a_{ik}}{1} = \frac{g}{h} \frac{a_i}{1} \in A_{\mathfrak{q}}.$$

$\Rightarrow$ $\exists\ l \notin \mathfrak{q}$ s.t. $l(a_{ik} h - g a_i) = 0$

$\Rightarrow$ $a_{ik}\, lh - a_i\, lg = 0.$

in the ring $\underline{A[l^{-1}, h^{-1}]}$ we have $\frac{a_{ik}}{1} = \frac{g}{h} \frac{a_i}{1}$