<u>Proof</u>: Injectivity means that there does not exist a rational function $f \in K(\mathbb{P}^n_k)$ s.t. $\mathrm{Div}(f)$ is a multiple of $[Z_0]$.

We study $\mathrm{Div}(f)$ for $f \in K = K(\mathbb{P}^n_k)$.

$$K = K(\mathbb{P}^n) = K(U_0) = \mathrm{Frac}\, k\left[\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right]$$
$$= k\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right)$$

<u>Claim</u>: $k\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right) = k(X_0, \dots, X_n)_0$,

<u>because</u>: we can write any f as $\frac{p}{q}$ where $p, q \in k\left[\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right]$ $\quad \frac{p}{q} = \dfrac{\frac{P}{X_0^r}}{\frac{Q}{X_0^s}}$ where P, Q are homogeneous of degree r and s respectively.

we can multiply P or Q by some power of X_0, hence assume $r = s$.

$\Rightarrow \quad f = \dfrac{P}{Q}$ where P, Q are homogeneous of the same degree (and coprime).

<u>Claim</u>: $\mathrm{Div}(f) = \mathrm{Div}(P) - \mathrm{Div}(Q)$

<u>proof</u>: Choose a point $x \in X = \mathbb{P}^n_k$ of codim. 1

$\exists\, i$ s.t. $x \in U_i = \mathrm{Spec}\, k[X_i^{-1}]_0 \subset X$

$x \leftrightarrow \mathfrak{p} \subset k[X_i^{-1}]_0$ prime of height 1

(comm. alg. from last quarter) $\Rightarrow \mathfrak{p}$ is principal

$\Rightarrow \quad \mathfrak{p} = \langle g \rangle \quad g$ irreducible pol. in $\dfrac{X_0}{X_i}, \dots, \dfrac{X_n}{X_i}$

$\Rightarrow \quad g = \dfrac{R}{X_i^d}$ where R is hom. of degree d in $X_0, \dots, X_n$

The local ring of x is $\mathcal{O}_{X,x} \cong k\left[\frac{X_0}{X_i}, \ldots, \frac{X_n}{X_i}\right]_{\mathfrak{p}}$

$$f = \frac{P}{Q} = \frac{\frac{P}{X_i^m}}{\frac{Q}{X_i^m}}$$ where m = degree of P = degree of Q

(recall $g = \frac{R}{X_i^d}$)

$$v_x(f) = v_{\mathfrak{p}}(f) = v_{\langle g \rangle}\left(\frac{P}{X_i^m}\right) - v_{\langle g \rangle}\left(\frac{Q}{X_i^m}\right)$$

= # times that g occurs as a factor of $\frac{P}{X_i^m}$

— # times that g occurs as a factor of $\frac{Q}{X_i^m}$

= # times that R divides P

— # times that R divides Q

$\Rightarrow$ $\mathrm{Div}(f) = \mathrm{Div}(P) - \mathrm{Div}(Q)$ $\square$

If f is constant, $\mathrm{Div}(f) = 0$

If f is not constant, then $\deg P = \deg Q > 0$

P and Q are coprime $\Rightarrow$ If $[Z_0]$ appears in $\mathrm{Div}(P)$, it will not appear in $\mathrm{Div}(Q)$ and vice-versa.

$\Rightarrow$ $\mathrm{Div}(f)$ cannot be just a multiple of $[Z_0]$. $\square$

Cartier divisors: We do not need $(*)$.

Here X is any noetherian scheme.

Def: The sheaf of total quotient rings $\mathcal{K}_X$:

Since affine open sets form a basis of the topology of X, we only define the sections of $\mathcal{K}_X$ on affine open sets.

Let $U = \operatorname{Spec} A \subset X$ be open affine, we define $\mathcal{K}_X(U)$ to be the total quotient ring of A, i.e., $\mathcal{K}_X(U)$ is the localization of A at all nonzerodivisors. Restriction maps are obtained from localization morphisms.

<u>Def</u>:
$$\begin{array}{ccc} \mathcal{K}_X^* & \subset & \mathcal{K}_X \\ \cup & & \cup \\ \mathcal{O}_X^* & \subset & \mathcal{O}_X \end{array}$$
$\mathcal{K}_X^* \subset \mathcal{K}_X$ is the subsheaf of invertible elements; $\mathcal{O}_X^* \subset \mathcal{O}_X$ " " " " " "

<u>Note</u>: $\mathcal{K}_X$ is a locally constant sheaf (exercise)

If X is integral, $\mathcal{K}_X$ is the constant sheaf with group K.

<u>Def</u>: A Cartier divisor on X is a global section of $\mathcal{K}_X^* / \mathcal{O}_X^*$.

What does the definition mean?

Consider the exact sequence (group law is multiplication)

$$1 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{K}_X^* \longrightarrow \mathcal{K}_X^*/\mathcal{O}_X^* \longrightarrow 1$$

Sections of $\mathcal{K}_X^*/\mathcal{O}_X^*$ can be locally lifted to $\mathcal{K}_X^*$:

$\forall \; f \in \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ $\exists$ covering $X = \bigcup_{i \in I} U_i$

s.t. $\forall i \quad \exists \; s_i \in \mathcal{K}_X^*(U_i)$ with $s_i \longmapsto f|_{U_i} \in \mathcal{K}_X^*/\mathcal{O}_X^*(U_i)$

and on $U_i \cap U_j$: $\left(s_i/s_j\right)\Big|_{U_i \cap U_j} \in \mathcal{O}_X^*(U_i \cap U_j)$

We call $\{(s_i, U_i)\}$ a representation of f. Note that it is not unique

Def: A Cartier divisor is called principal if it is in the image of the natural map

$$\Gamma(X, \mathcal{K}_X^*) \longrightarrow \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*).$$

In other words it has a representation of the form

$$\{(s, X)\}.$$

Def: Two Cartier divisors are called linearly equivalent if their difference (for the multiplicative group structure) is principal. In other words, they have representations of the form $\{(s_i, U_i)\}$ and $\{(s s_i, U_i)\}$

Relation with invertible sheaves: Given a representation $\{(f_i, U_i)\}$ of a Cartier divisor D, we define an invertible subsheaf $\mathcal{O}_X(D)$ of $\mathcal{K}_X$ as follows.

On each U_i, define $\mathcal{O}_X(D)|_{U_i} := \mathcal{O}_{U_i} \cdot f_i^{-1} \subset \mathcal{K}_{U_i}$

These glue to a subsheaf $\mathcal{O}_X(D)$ of $\mathcal{K}_X$ because on $U_i \cap U_j$:

$$\mathcal{O}_{U_i \cap U_j} f_i^{-1}|_{U_i \cap U_j} = \mathcal{O}_{U_i \cap U_j} f_j^{-1}|_{U_i \cap U_j} \subset \mathcal{K}_{U_i \cap U_j}$$

because $f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$

Check (exercise) $\mathcal{O}_X(D)$ <u>does not</u> depend on the choice of representation.

Conversely, given an invertible subsheaf $\mathcal{L}$ of $\mathcal{K}_X$, we can associate a Cartier divisor D to it such that $\mathcal{O}_X(D) = \mathcal{L}$ as follows.

Choose an open covering $X = \bigcup_{i \in I} U_i$ s.t. $\forall i$ $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$.

$\forall i$, define f_i^{-1} as the image of 1:

$$\begin{array}{ccccc} \mathcal{O}_{U_i} & \xrightarrow[\varphi_i]{\cong} & \mathcal{L}|_{U_i} & \hookrightarrow & \mathcal{K}_{U_i} \\ \psi & & & & \psi \\ 1 & & \longmapsto & & f_i^{-1} \end{array}$$

On $U_i \cap U_j$, we have $f_i / f_j \in \mathcal{O}_X^*(U_i \cap U_j)$

because $f_i^{-1}|_{U_i \cap U_j}$ and $f_j^{-1}|_{U_i \cap U_j}$ generate the same trivial $\mathcal{O}_{U_i \cap U_j}$-submodule of $\mathcal{K}_{U_i \cap U_j}$.

So $\{(f_i, U_i)\}$ represents a Cartier divisor D, and, by def., $\mathcal{O}_X(D) = \mathcal{L}$.

<u>So</u>: Cartier divisors are in bijection with invertible subsheaves of $\mathcal{K}_X$.