

Def: The (Weil) divisor of a rational function:

$f \in K = K(X)$   $X$  satisfies  $(*)$ ,  $K(X) =$  field of rational functions

$$\text{Div}(f) := \sum_{\substack{Y \subset X \\ \text{integral Weil divisor}}} v_Y(f) [Y]$$

$v_Y(f) \in \mathbb{Z}$  is the valuation of  $f$  for the DVR,  $\mathcal{O}_{Y,\eta}$   
 $\eta =$  generic point of  $Y$ .

$$K = \text{Frac}(\mathcal{O}_{Y,\eta}) \quad \exists u \in \mathcal{O}_{Y,\eta}^*$$

$$\text{s.t. } f = u \pi^{v_Y(f)}$$

where  $\pi$  is a uniformizer for  $\mathcal{O}_{Y,\eta}$

The divisors of rational functions are called principal divisors.

$$\forall f, g \in K \quad v_Y(fg^{-1}) = v_Y(f) - v_Y(g)$$

$\Rightarrow$  principal divisors form a subgroup of  $\text{Div}(X)$ .

Def:  $\mathcal{C}l(X) := \text{Div}(X) / \text{Prin}(X)$

the class  
group of  $X$

$\uparrow$  principal divisors

We need to verify that  $\sum v_Y(f)[Y]$  is finite to

make sense.  $\text{Div}(f)$  is well-defined.

Lemma:  $\forall f \in K$ ,  $v_Y(f) = 0$  for all but finitely many integral Weil divisors  $Y$ .

Proof: Let  $U = \text{Spec } A \subset X$  be open and  $\neq \emptyset$ .

Then  $K = \text{Frac}(A) \Rightarrow \exists g, h \in A$  s.t.  $f = \frac{g}{h}$

After replacing  $A$  by  $A[h^{-1}]$ , we can assume  $f \in A$ .

$\Rightarrow \forall p \in \text{Spec} A$  of height 1,  $f \in A_p = \mathcal{O}_{X,p}$

$$\Rightarrow v_p(f) \geq 0$$

$\Rightarrow$  the set of  $\eta \in X$  s.t.  $v_\eta(f) < 0$  is contained

in  $X \setminus U =: Y$   $Y \subset X$  is closed.

$X$  is noetherian  $\Rightarrow Y =$  finite union of irreducible components  $= Y_1 \cup \dots \cup Y_n$

$$U \neq \emptyset \Rightarrow \forall i, Y_i \neq X \Rightarrow \text{codim } Y_i \geq 1$$

$\Rightarrow v_\eta(f) < 0 \Rightarrow \eta$  is the generic point of one of the  $Y_i$

$\Rightarrow$  there are at most finitely many  $\eta$  with  $v_\eta(f) < 0$ .

replace  $f$  with  $\frac{1}{f} \Rightarrow$  there are at most finitely many  $\eta$  with  $v_\eta(f) > 0$ .  $\square$

Def: We say two Weil divisors  $D_1, D_2 \in \text{Div}(X)$  are linearly equivalent if  $\exists f \in K$  s.t.

$$D_1 - D_2 = \text{Div}(f),$$

i.e.,  $D_1, D_2$  have the same image in  $\mathcal{C}(X)$ .

We also say  $D_1, D_2$  have the same linear equivalence class.

Prop.: If  $X = \text{Spec} A$  (in particular  $A$  is a noetherian integral domain), then  $A$  is a UFD iff  $X$  is normal (i.e.,  $A$  is integrally closed) and  $\mathcal{C}(X) = 0$ .

Proof: (see Hartshorne).

Corollary: If  $X = \mathbb{A}_k^n$ , then  $\mathcal{O}(X) = 0$

Divisors in projective space  $X = \mathbb{P}_k^n = \text{Proj } S$

$$S := k[X_0, \dots, X_n] = \bigoplus_{d \geq 0} S_d$$

recall  $S_d = \Gamma(\mathcal{O}_X(d))$

For  $s \in S_d$ , put  $Y := Z(s)$

Lemma:  $Y$  is a closed subscheme of pure codimension 1 of  $X$  (i.e., all the irreducible components of  $Y$  have codim. 1).

The irreducible components of  $Y$  are the zeros of the irreducible factors of  $s$ . Furthermore, the homogeneous ideal  $I_Y$  is generated by  $s$ .

Proof: The ideal sheaf  $\mathcal{I}_Y$  is the image of

$$s: \mathcal{O}_X(-d) = \mathcal{O}_X(d)^* \longrightarrow \mathcal{O}_X.$$

In each open set  $U_i := \text{Spec } S[x_i^{-1}]_0 = \text{Spec } k\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]$

$$\mathcal{O}_X(d) = \mathcal{O}_{U_i} \cdot x_i^d \quad \mathcal{O}_X(-d) = \mathcal{O}_{U_i} \cdot x_i^{-d}$$

$$s|_{U_i}: \mathcal{O}_{U_i} \cdot x_i^{-d} \xrightarrow{s} \mathcal{O}_{U_i}$$

$$\searrow \mathcal{O}_{U_i} \cdot \frac{s}{x_i^d}$$

$$\frac{s}{x_i^d} \in k\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]$$

$$\Rightarrow \mathcal{I}_Y|_{U_i} = \Gamma(\mathcal{I}_Y|_{U_i}) = \left\langle \frac{s}{x_i^d} \right\rangle \subset S[x_i^{-1}]_0$$

The homogeneous ideal  $\mathcal{I}_Y = \bigoplus_{e \geq 0} \mathcal{I}_Y|_e \subset S$

$$I_{\gamma, e} = \{t \in S_d \mid \mathcal{D}_{z(t)} \subset \mathcal{D}_\gamma\}$$

$$\text{on } U_i : \mathcal{D}_{z(t) \cap U_i} \subset \mathcal{D}_{\gamma \cap U_i}$$

$$\Leftrightarrow I_{z(t) \cap U_i} \subset I_{\gamma \cap U_i} = \left\langle \frac{s}{x_i^d} \right\rangle$$

$$\parallel$$

$$\left\langle \frac{t}{x_i^e} \right\rangle$$

$$\Rightarrow \frac{t}{x_i^e} \in \left\langle \frac{s}{x_i^d} \right\rangle \quad \forall i = 0, \dots, n$$

(exercise)  $\Rightarrow$   $t$  is a multiple of  $s$ , i.e.,  $t \in \langle s \rangle \subset S$

$$\Rightarrow I_\gamma = \langle s \rangle \subset S$$

Now write  $s = s_1^{m_1} \cdots s_l^{m_l}$

where  $s_i$  is irreducible (prim.) of degree  $d_i$  and  $s_i, s_j$  are not proportional for  $i \neq j$ .

$\Rightarrow \mathcal{I}_{Z(s)} = \text{product of } \mathcal{I}_{Z(s_i)}^{m_i} \text{ in } \mathcal{O}_X$

$\Rightarrow Z(s) = \bigcup_{\text{as a set } i=1}^l Z(s_i)$

(recall that  $Z(s) \stackrel{\text{as a set}}{=} \left\{ x \in X \mid s(x) \in \mathfrak{m}_x \subset \mathcal{O}_{X,x} \right\}$   
 $= \bigcup_{i=1}^l \left\{ x \in X \mid s_i(x) \in \mathfrak{m}_x \right\}$ )

$I_{Z(s_i)} = \langle s_i \rangle$

$\Rightarrow Z(s_i)$  is irreducible of codimension 1 in  $X$ .  
 (from last quarter: intersect with  $V_j$   
 s.t.  $Z(s_i) \cap V_j \neq \emptyset$ )

Def. The Weil divisor of  $s$  is  $\text{Div}(s) = \sum_{i=1}^l m_i [Z(s_i)]$  □

Next we wish to determine  $\mathcal{C}l(\mathbb{P}_k^n)$ .

We first need a general lemma (works for all  $X$  satisfying  $(*)$ ).



General lemma:  $U \subset X$  open  $\neq \emptyset$ , let

$Z_1, \dots, Z_n$  be the codimension 1 irreducible components of  $X \setminus U$ . Then, intersecting divisors of  $X$  with  $U$  produces the exact sequence

$$0 \rightarrow \mathbb{Z}[Z_1] \oplus \dots \oplus \mathbb{Z}[Z_n] \rightarrow \text{Div}(X) \rightarrow \text{Div}(U) \rightarrow 0$$

$D \longmapsto D \cap U$

which induces the exact sequence

$$\mathbb{Z}[Z_1] \oplus \dots \oplus \mathbb{Z}[Z_n] \rightarrow \mathcal{C}(X) \rightarrow \mathcal{C}(U) \rightarrow 0.$$

Proof: The first exact sequence is immediate: a divisor

of  $U$  is the intersection with  $U$  of its closure in  $X$ .

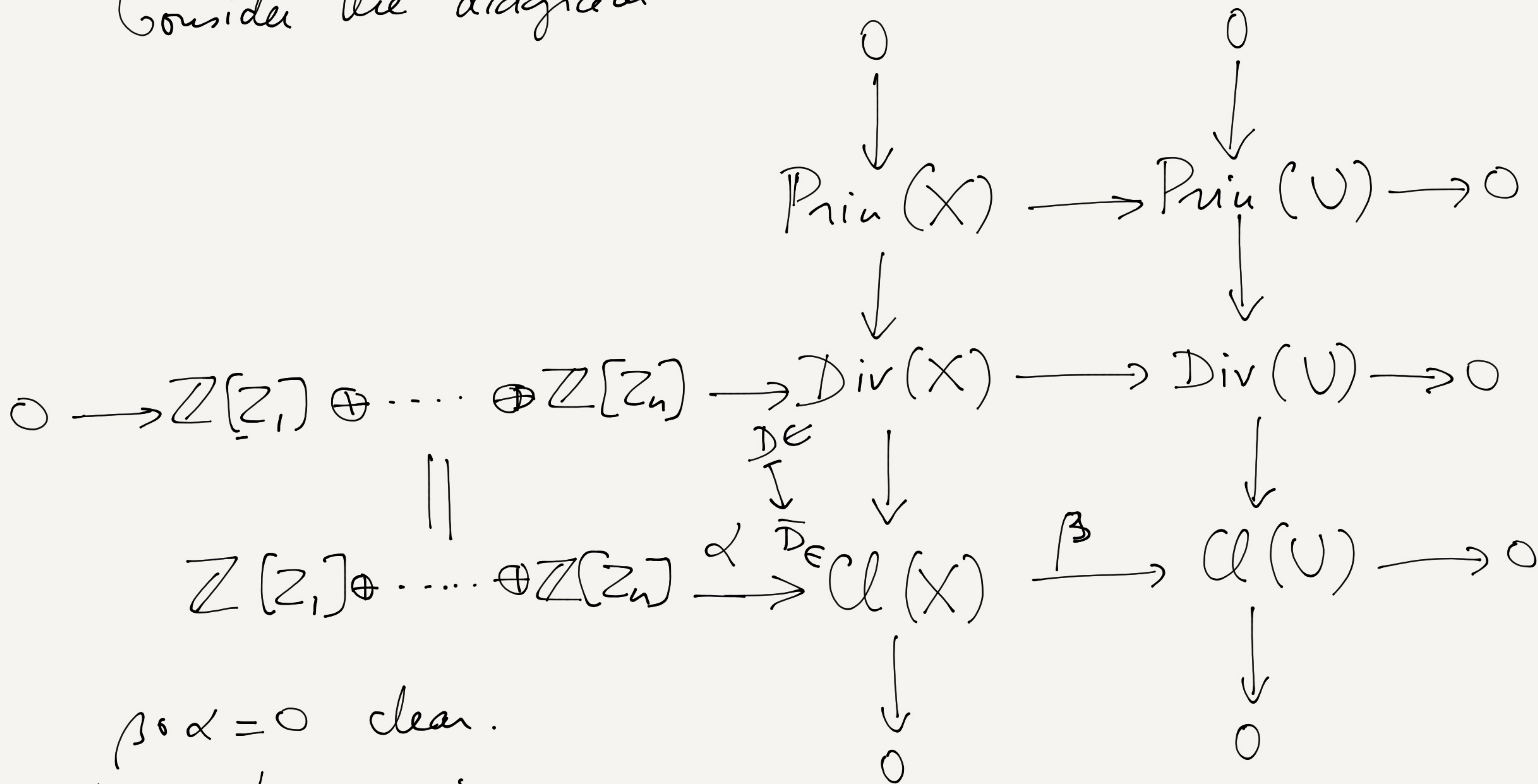
On the second sequence, surjectivity is true for the same reason.

We prove exactness in the middle:

Note:  $K = K(X) = K(U)$

for  $f \in K$   $\text{Div}_X(f) \cap U = \text{Div}_U(f)$ .

Consider the diagram



$\beta \circ \alpha = 0$  clear.

show  $\ker \beta \subset \text{im } \alpha$

If  $D \in \text{Div}(X)$  has image in  $\ker \beta$ , this means

$$\exists f \in K \text{ s.t. } D \cap U = \text{Div}_U(f).$$

$$\Rightarrow (D - \text{Div}_X(f)) \cap U = 0$$

$\Rightarrow D - \text{Div}_X(f)$  supported on  $X \setminus U$

$$\Rightarrow D - \text{Div}_X(f) \in \mathbb{Z}[z_1] \oplus \dots \oplus \mathbb{Z}[z_n]$$

$$\Rightarrow \bar{D} = \overline{(D - \text{Div}_X(f))} \in \mathcal{C}(X)$$

$\in$  image of  $\mathbb{Z}[z_1] \oplus \dots \oplus \mathbb{Z}[z_n]$ .

$\Rightarrow$  exactness in the middle.



We apply the lemma to compute  $\mathcal{C}(\mathbb{P}_k^n)$ :

$$\text{take } U = U_0 = \text{Spec } k\left[\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right]$$

$$Z_0 = \mathbb{P}_k^n \setminus U_0 \text{ is irreducible}$$

$$= Z(X_0) \quad X_0 \in S_1 = \Gamma(G_X(1))$$

$$\mathbb{Z}[Z_0] \longrightarrow \mathcal{C}(\mathbb{P}_k^n) \longrightarrow \mathcal{C}(U_0) \longrightarrow 0$$

$$U_0 \cong \mathbb{A}_k^n \implies \mathcal{C}(U_0) = 0$$

$$\implies \mathbb{Z}[Z_0] \longrightarrow \mathcal{C}(\mathbb{P}_k^n)$$

Lemma: The map  $\mathbb{Z}[Z_0] \longrightarrow \mathcal{C}(\mathbb{P}_k^n)$  is injective, i.e.,

$$\mathbb{Z}[Z_0] \xrightarrow{\cong} \mathcal{C}(\mathbb{P}_k^n)$$