Proof of the main lemma:

We have the exact sequence

\[ 0 \rightarrow I_Y \rightarrow S \rightarrow S(Y) \rightarrow 0 \]

⇒ (A) \[ 0 \rightarrow I_Y \rightarrow S \rightarrow S(Y) \rightarrow 0 \]

and the exact sequence

(B) \[ 0 \rightarrow I_Y \rightarrow \tilde{S} \rightarrow \tilde{S}(Y) \rightarrow 0 \]

where \( i : Y \hookrightarrow X = \mathbb{P}^n_R = \text{Proj} \ S \).

First, by the fact that \( \mathcal{F} \rightarrow \mathcal{G} \) for coherent sheaves \( \mathcal{F} \) on \( X \), we have \( I_Y \subset \mathcal{I}_Y \) and \( \tilde{S} \cong \mathcal{O}_X \)

and \( I_Y \hookrightarrow \mathcal{S} \) is the global sections of \( \mathcal{S} \rightarrow \mathcal{O}_X \)

⇒ we have the commutative diagram:

\[
\begin{array}{ccc}
I_Y & \hookrightarrow & \tilde{S} \\
\downarrow & & \downarrow \\
\mathcal{I}_Y & \hookrightarrow & \mathcal{O}_X
\end{array}
\]
we can form the commutative diagram with exact rows

(A) \[ 0 \to \tilde{I}_Y \to \tilde{S} \to \tilde{S}(Y) \to 0 \]

(B) \[ 0 \to \tilde{I}_Y \to \tilde{O}_X \to \tilde{O}_Y \to 0 \]

First consequence: \[ \text{Support}(\tilde{S}(Y)) = \text{Support}(\tilde{I}_Y) = \emptyset \]

Claim: \[ \text{Support}(\tilde{S}(Y)) = \{ \mu \in X \mid \tilde{S}(Y)\mu \neq 0 \} \]

Proof: \[ \text{Support}(\tilde{S}(Y)) = \{ \mu \in X \mid \tilde{S}(Y)\mu \neq 0 \} \]
\[ = \{ \mu \in X \mid (\tilde{S}(Y)\mu)_0 \neq 0 \} \]
\[ = \{ \mu \in X \mid S(Y)\mu \neq 0 \} \]
\[ = \{ \mu \in X \mid (\tilde{S}(Y)\mu)_0 \neq 0 \} \]
\[ \text{Now: } (S/I_{\gamma})_{\rho} = 0 \iff S_{\rho} = (I_{\gamma})_{\rho} \]
\[ \iff I_{\gamma} \setminus \rho \neq \emptyset \]
\[ \iff I_{\gamma} \notin \rho \]

\[ \text{hence } \text{Support} (S_{\tilde{\gamma}}) = \{ \rho \in X \mid I_{\gamma} \subset \rho \} \]
\[ \implies Y = \text{Support} (S_{\tilde{\gamma}}) = \{ \rho \in X \mid \rho \supset I_{\gamma} \} \]
\[ = \text{Proj} (S(Y)) \leftarrow \text{Proj} \tilde{S} \text{ natural} \]

\[ \text{and } \xi_{\gamma} Y \cong S_{\tilde{\gamma}} \cong \xi_{\gamma} \circ \text{Proj} (S(Y)) \]

\[ \Rightarrow Y = \text{Proj} S(Y) \leftarrow \text{Proj} \tilde{S} \]
Divisors: The Zariski topology is coarse.

Algebraic subvarieties of an algebraic variety carry a lot of information about the ambient variety. So we want to study them. The simplest of these are those that have codimension 1: these give rise to the notion of Weil divisors. The ideal sheaves of codim. 1 subvarieties give rise to the notion of Cartier divisors.

These also help us understand morphisms from one scheme to another: we can pull back Cartier divisors, we can push forward Weil divisors. Cartier divisors are closely related to invertible sheaves.
From now on, we will assume $X$ is an integral and noetherian scheme, separated over $\text{Spec } \mathbb{Z}$.

**Def.** An integral or prime Weil divisor is a closed subscheme $Y$ of $X$ which is integral of codimension $1$.

**Note.** $Y$ is the closure of its generic point, say $y \in X$.

**Lemma.** The local ring $\mathcal{O}_{Y, y}$ has dimension $1$.

**Proof.** Choose an open affine $\text{Spec } A \subset X$ s.t.

$Y \cap \text{Spec } A \neq \emptyset \implies y \in \text{Spec } A$

$Y \cap \text{Spec } A \subset \text{Spec } A$ is a closed subscheme, integral and of codimension $1$.

$y \in p \subset A$ prime of height $1$. 

\[ 0 \rightarrow \mathfrak{p} \rightarrow A \rightarrow A/\mathfrak{p} \rightarrow 0 \]

and localize at \( \mathfrak{p} \):

\[ 0 \rightarrow \mathfrak{p}A_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}} \rightarrow (A/\mathfrak{p})_{\mathfrak{p}} \rightarrow 0 \]

local ring \( \mathbb{Q}(x,y) \)

residue field of \( \mathfrak{p} \)

any chain of prime ideals of \( A_{\mathfrak{p}} \) is contained in

\( \mathfrak{p}A_{\mathfrak{p}} \) \implies only one chain \( 0 < \mathfrak{p}A_{\mathfrak{p}} \) (\( \mathfrak{p} \) has height 1)

\( \Rightarrow A_{\mathfrak{p}} \) has dim. 1.

So integral Weil divisors in \( X \) are in 1-to-1 correspondence with the points of \( X \) whose local rings have dim 1.

Def: A Weil divisor on \( X \) is a formal linear combination of integral Weil divisors with integer coefficients. The set of all Weil divisors is the free abelian group with basis
all the integral Weil divisors:

\[ \text{Div}(X) := \left\{ \sum_{\text{finite}} n_Y [Y] \quad | \quad n_Y \in \mathbb{Z}, \quad Y \subseteq X \text{ integral Weil div.} \right\} \]

Example: If \( X \) has dimension 1, then integral Weil divisors are the closed points of \( X \). Weil divisors are of the form \( \sum_{\text{closed pt}} n_P [P] \).

We need to impose more restriction on \( X \), in order to have a nice relation between Weil and Cartier divisors. We need a fundamental result in commutative algebra: Nakayama's lemma.
Def: The Jacobson radical $J(A)$ of a ring $A$ is the intersection of all the maximal ideals of $A$.

Proposition (Nakayama's lemma): Let $0 \subseteq A$ be an ideal, suppose $0 \subseteq J(A)$. For any finitely generated $A$-module, if $0_0 M = M$, then $M = 0$.

Consequence 1: If $N \subseteq M$ is a submodule, $M = 0_0 M + N \implies M = N$ (apply the prop. to $M/N$).

Consequence 2: If $A$ is a local ring with maximal ideal $M$, and $x_1, \ldots, x_n$ are elements of $M$, then if the images of $x_1, \ldots, x_n$ in $M/M$ generate it, $x_1, \ldots, x_n$ generate $M$. 
We will have a nice relation between Weil and
Cartier divisors when $X$ is “regular in codim. 1”.
This means all the 1-dim. local rings of $X$ are “regular”

**Def**: A finite dimensional noetherian local ring with
maximal ideal $M$ is regular if $\dim R$ is equal to the
minimal number of generators of $M$.

Equivalently, by Consequence 2, $\dim R = \dim_ k M / M^2$
where $k = R / M$

$(M / M^2 \cong M \otimes R / M \cong \frac{M}{R})$

**Def**: A discrete valuation ring (DVR) is a noetherian
regular local ring of dim. 1.
Facts: \( \text{If } R \text{ is a DVR, then } \mathcal{M} \text{ is principal.} \)

A generator of \( \mathcal{M} \) is called a uniformizer.

Any ideal of \( R \) is a power of \( \mathcal{M} \Rightarrow R \text{ is a PID.} \)

For any \( f \in R \), \( \exists \) \( m \geq 0 \) and \( u \in R^\times \) (unit)

\[ s.t. \quad f = um + t^n \]

If \( K := \text{Frac}(R) \), then \( \forall \ g \in K, \exists \ u \in R^\times \)
and \( m \in \mathbb{Z} \) s.t. \( g = um + t^n \).

The map \( g \mapsto \nu_R(g) := n \) is a valuation of \( K \)

with valuation ring \( \mathcal{R} \): this is a discrete valuation.

Hypothesis(\( \star \)): \( X \) is noetherian, integral, separated, regular in codim 1.
In particular, all the 1-dim. local rings of \( X \) are DVRs.

We define an equivalence relation on Weil divisors.

Let \( K \) be the function field of \( X \); this is the local ring of the generic point of \( X \) and the field of fractions of the ring \( A \) of any affine open \( \text{Spec} \mathcal{A} \subset X \).

We refer to the elements of \( K \) as rational functions on \( X \).