Remark: For any $\mathcal{O}_X$-module $\mathcal{F}$ and collection $\{s_i \mid i \in I\} \subset \Gamma(X, \mathcal{F})$, we can define a morphism of $\mathcal{O}_X$-modules: $\varphi : \mathcal{O}_X^{\oplus I} \longrightarrow \mathcal{F}$ which on any open $U \subset X$ sends $\sum_{i \in I} f_i \in (\mathcal{O}_X^{\oplus I})(U)$ to

$$\sum_{i \in I} f_i \, s_i|_U \in \mathcal{F}(U).$$

We have that $\mathcal{F}$ is generated by $\{s_i \mid i \in I\}$ iff $\varphi$ is surjective.

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Another notion we need: Zeros of sections of (quasi-)coherent sheaves.

Let $\mathcal{F}$ be a (quasi-)coherent sheaf on a noetherian scheme $X$ and $s \in \Gamma(X, \mathcal{F})$. We define the scheme of zeros of $s$, denoted $Z(s)$, as follows (this is a closed subscheme of $X$):

$s$ defines a morphism of $\mathcal{O}_X$-modules

$$s : \mathcal{F}^* \longrightarrow \mathcal{O}_X$$

on any $U \subset X$ open $\mathcal{F}^*(U) \ni \ell \longmapsto \ell(s|_U) \in \mathcal{O}_X(U)$

$$\ell \in \mathcal{F}^*(U) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{O}_U) \qquad \mathcal{O}_X(U) = \mathcal{O}_U(U)$$

The image of $s$ is a coherent $\mathcal{O}_X$-submodule of $\mathcal{O}_X$, i.e., a coherent sheaf of ideals on $X$; the associated closed subscheme is, by definition, $Z(s)$.

<u>Claim 1</u>: The support $\mathrm{Supp}\, Z(s)$ of $Z(s)$ (i.e., the underlying closed subset of $Z(s)$) is the set of points $x \in X$ s.t. the image of $s_x : \mathcal{F}_x^* \to \mathcal{O}_{X,x}$ is contained

in the maximal ideal $m_x \subset \mathcal{O}_{X,x}$.

Proof:
$$\begin{tikzcd} s: \mathcal{F}^* \arrow[r] \arrow[d, two heads] & \mathcal{O}_X \\ \mathcal{I}_{Z(s)} \arrow[ur, hook] & \end{tikzcd}$$

$$0 \to \mathcal{I}_{Z(s)} \to \mathcal{O}_X \to i_* \mathcal{O}_{Z(s)} \to 0$$

where $i: Z(s) \hookrightarrow X$

$$\begin{aligned} x \in Z(s) &\iff (i_* \mathcal{O}_{Z(s)})_x \neq 0 \\ &\iff \mathcal{I}_{Z(s),x} \not\subseteq \mathcal{O}_{X,x} \\ &\iff \mathcal{I}_{Z(s),x} \subset m_x \subset \mathcal{O}_{X,x} \\ &\iff s(\mathcal{F}_x^*) \subset m_x . \end{aligned}$$

Claim 2: Suppose $\mathcal{F}$ is locally free of finite rank $r$.
Then, for $x \in X$, $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module of rank $r$:

$$\mathcal{F}_x \cong \mathcal{O}_{X,x}^{\oplus n}$$

$$\Rightarrow \quad \mathcal{F}_x^* = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{O}_{X,x}) \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}^{\oplus n}, \mathcal{O}_{X,x})$$

$$\cong \mathcal{O}_{X,x}^{\oplus n}$$

The condition $s_x(\mathcal{F}_x^*) \subset \mathfrak{m}_x$ means $\boxed{s(x) \in \mathfrak{m}_x \mathcal{F}_x.}$

Indeed: if we pass to the quotient by $\mathfrak{m}_x$:

$$\mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x \cong \mathcal{O}_{X,x}^{\oplus n} / \mathfrak{m}_x^{\oplus n} \cong k(x)^{\oplus n}$$

and $$\mathcal{F}_x^* / \mathfrak{m}_x \mathcal{F}_x^* \cong \left( \mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x \right)^* \cong \left( k(x)^{\oplus n} \right)^*$$

$s_x(\mathcal{F}_x^*) \subset \mathfrak{m}_x \iff s_x$ induces the $0$ linear map on $(k(x)^{\oplus n})^*$

$\iff s_x = 0 \in k(x)^{\oplus n} = \mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x$

$\iff s(x) \in \mathfrak{m}_x \mathcal{F}_x.$

Note: This fails if $\mathcal{F}$ is not locally free:

e.g.: $\mathcal{F}$ = skyscraper sheaf supported on a proper closed subset of $X$.

Then $\mathcal{F}^* = O_X$ the zero sheaf on $X$

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## Homogeneous ideals of subsets of projective space:

Assume $S := R[X_0, \dots, X_n]$, $R$ comm. ring

$X := \operatorname{Proj} S = \mathbb{P}^n_R$

We saw that $\Gamma(X, \mathcal{O}_X(d)) = S_d \quad \forall d \in \mathbb{Z}$

i.e., $\Gamma_*(\mathcal{O}_X) = S$.

Def: (1) For a subset $Y \subset X$, define

$$I_{Y,d} := \{ s \in S_d \mid Y \subset Z(s) \}$$

= the set of homogeneous polynomials of degree $d$ vanishing on $Y$ $:= \{ s \in S_d \mid \forall x \in Y,\ s(x) \in m_x \mathcal{O}_{X,x}(d) \cong m_x \}$

(2) For a closed subscheme $Y \subset X$ with ideal sheaf $\mathcal{I}_Y$, define

$$I_{Y,d} := \{ s \in S_d \mid \mathcal{I}_{Z(s)} \subset \mathcal{I}_Y \}$$
$$= \{ s \in S_d \mid Y \subset Z(s) \text{ as subschemes} \}$$

<u>Ex</u>: If $Y$ is reduced, the two definitions are equal.

<u>Def</u>: The homogeneous ideal of a subset or closed subscheme $Y \subset X = \mathbb{P}^n_R$ is

$$I_Y := \bigoplus_{d \in \mathbb{Z}} I_{Y,d} \subset S$$

The homogeneous coordinate ring of $Y$ is $S(Y) := S / I(Y)$.

**Remark:** Choose $s \in S_d$ and suppose $Y \subset X$ is a closed subscheme. We have $s \in I_{Y,d}$ iff $\mathcal{I}_{Z(s)} \subset \mathcal{I}_Y$.

$\mathcal{I}_{Z(s)}$ is the image of $s : \mathcal{O}_X(d)^* \longrightarrow \mathcal{O}_X$

recall that $\mathcal{O}_X(d)^* \cong \mathcal{O}_X(-d)$, so

$\mathcal{I}_{Z(s)}$ is the image of $s : \mathcal{O}_X(-d) \longrightarrow \mathcal{O}_X$

$\mathcal{I}_{Z(s)} \subset \mathcal{I}_Y \iff$

$$\begin{tikzcd} s : \mathcal{O}_X(-d) \arrow[rr] \arrow[dr] & & \mathcal{O}_X \\ & \mathcal{I}_Y \arrow[ur, hook] & \end{tikzcd}$$

factors

twist by $\mathcal{O}_X(d)$ :

$$\begin{tikzcd} \mathcal{O}_X(-d) \otimes \mathcal{O}_X(d) \arrow[rr, "s"] \arrow[dr] & & \mathcal{O}_X(d) \\ & \mathcal{I}_Y(d) \arrow[ur, hook] & \end{tikzcd}$$

$$\iff \begin{tikzcd} \mathcal{O}_X \arrow[rr, "s"] \arrow[dr] & & \mathcal{O}_X(d) \\ & \mathcal{I}_Y(d) \arrow[ur] & \end{tikzcd}$$

**Fact:** $\mathcal{O}_X \xrightarrow{s} \mathcal{O}_X(d)$ is "multiplication" by $s$:

$$\forall\, U \subset X \qquad \begin{array}{ccc} \mathcal{O}_X(U) & \xrightarrow{s} & \mathcal{O}_X(d)(U) \\ \rotatebox{90}{$\in$} & & \\ f & \longmapsto & f\, s|_U \in \mathcal{O}_X(d)(V) \end{array}$$

$s$ factors through $\mathcal{I}_Y(d) \iff s \in \Gamma(X, \mathcal{I}_Y(d))$

**More generally:** $\forall$ $\mathcal{O}_X$-module $\mathcal{F}$,

$$\begin{array}{ccc} \Gamma(X, \mathcal{F}) & = & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \\ \rotatebox{90}{$\in$} & & \\ s & \longmapsto & (f \longmapsto f\, s|_U \text{ on any } U) \\ \varphi(1) & \longleftarrow & \varphi \end{array}$$

---

$\Rightarrow \quad s \in \Gamma(X, \mathcal{I}_Y(d)) \iff s : \mathcal{O}_X \to \mathcal{I}_Y(d)$

**Conclusion:** $s \in I_{Y,d} \iff s \in \Gamma(X, \mathcal{I}_Y(d))$.

i.e., $I_{Y,d} = \Gamma(X, \mathcal{I}_Y(d))$.

Note that $\mathcal{I}_Y \hookrightarrow \mathcal{O}_X \Rightarrow \mathcal{I}_Y(d) \hookrightarrow \mathcal{O}_X(d)$

$$\Rightarrow \Gamma(\mathcal{I}_Y(d)) \hookrightarrow \Gamma(\mathcal{O}_X(d)) \hookleftarrow I_{Y,d}$$

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$$\Rightarrow \quad I_Y = \bigoplus_{d \in \mathbb{Z}} I_{Y,d} = \bigoplus_{d \in \mathbb{Z}} \Gamma(\mathcal{I}_Y(d))$$

$$=: \Gamma_*(\mathcal{I}_Y)$$

Recall: If $S$ is finitely generated by $S_1$ as an $S_0$-alg. and $\mathcal{F}$ is quasi-coherent, then $\widetilde{\Gamma_*(\mathcal{F})} \xrightarrow{\cong} \mathcal{F}$

This is true for $S = R[X_0, \dots, X_n]$ and $\mathcal{F} = \mathcal{I}_Y$,

so $\widetilde{\Gamma_*(\mathcal{I}_Y)} \xrightarrow{\cong} \mathcal{I}_Y$, i.e., $\widetilde{I_Y} \xrightarrow{\cong} \mathcal{I}_Y$.

<u>Main</u> <u>Lemma</u>: There is a natural isomorphism

$$g : Y \xrightarrow{\cong} \operatorname{Proj} S(Y)$$

<u>Proof</u>: The quotient morphism $\varphi : S \twoheadrightarrow S(Y) = S/I(Y)$ induces a morphism $f : \operatorname{Proj} S(Y) \longrightarrow \operatorname{Proj} S$.

($f$ is defined everywhere because $\varphi$ is surjective)

We will show that $f$ factors through an isom.

$$\operatorname{Proj} S(Y) \xrightarrow{\cong} Y \hookrightarrow X := \operatorname{Proj} S$$

We have the exact sequence

$$0 \to I_Y \longrightarrow S \longrightarrow S(Y) \longrightarrow 0 \text{ (by def.)}.$$

<u>Lemma 1</u>: The $\sim$ functor is exact.

Proof: Suppose given an exact sequence of graded $S$-modules: $0 \to M' \to M \to M'' \to 0$

$\forall d$, $\forall\ t \in S_d$, we have the exact sequences

$$0 \to M'[t^{-1}] \to M[t^{-1}] \to M''[t^{-1}] \to 0$$

and $0 \to M'[t^{-1}]_0 \to M[t^{-1}]_0 \to M''[t^{-1}]_0 \to 0$

on the affine scheme $U_{(t)} := \operatorname{Spec} S[t^{-1}]_0 \subset X$

$$\Rightarrow \quad 0 \to \widetilde{M'[t^{-1}]_0} \to \widetilde{M[t^{-1}]_0} \to \widetilde{M''[t^{-1}]_0} \to 0$$

on $U_{(t)}$

$$\|$$

$$0 \to \tilde{M}'\big|_{U_{(t)}} \to \tilde{M}\big|_{U_{(t)}} \to \tilde{M}''\big|_{U_{(t)}} \to 0$$

The open sets $U_{(t)}$ cover $X$ $\Rightarrow$ $0 \to \tilde{M}' \to \tilde{M} \to \tilde{M}'' \to 0$ is exact on $X$. $\square$