Remark: For any \( \mathcal{O}_X \)-module \( \mathcal{F} \) and collection \( \{ s_i \mid i \in I \} \subset \mathcal{F}(X, \mathcal{F}) \), we can define a morphism of \( \mathcal{O}_X \)-modules: \( \varphi \colon \mathcal{O}_X \otimes I \to \mathcal{F} \)

which on any open \( U \subset X \) sends \( \sum_{i \in I} f_i \in (\mathcal{O}_X)(U) \) to \( \sum_{i \in I} f_i s_i |_U \in \mathcal{F}(U) \).

We have that \( \mathcal{F} \) is generated by \( \{ s_i \mid i \in I \} \) iff \( \varphi \) is surjective.

Another notion we need: Zeros of sections of (quasi-)coherent sheaves.

Let \( \mathcal{F} \) be a (quasi-)coherent sheaf on a noetherian scheme \( X \) and \( s \in \Gamma(X, \mathcal{F}) \). We define the scheme of zeros of \( s \),
denoted $Z(s)$, as follows (this is a closed subscheme of $X$): $s$ defines a morphism of $\mathcal{O}_X$-modules

$$s : \mathcal{F}_s^* \to \mathcal{O}_X$$

on any $U \subseteq X$ then $\mathcal{F}_s^*(U) \ni s_U \mapsto l(s_U) \in \mathcal{O}_X(U)$

$$l \in \mathcal{F}_s^*(U) = \text{Hom}_{\mathcal{O}_U}(\mathcal{F}_s|_U, \mathcal{O}_U) \subseteq \mathcal{O}_U(U)$$

The image of $s$ is a coherent $\mathcal{O}_X$-submodule of $\mathcal{O}_X$, i.e., a coherent sheaf of ideals in $X$, the associated closed subscheme is, by definition, $Z(s)$.

Claim 1: The support $\text{Supp } Z(s)$ of $Z(s)$ (i.e., the underlying closed subset of $Z(s)$) is the set of points $x \in X$ s.t. the image of $s_x : \mathcal{F}_s^* \to \mathcal{O}_X, x$ is contained
in the maximal ideal $M_x \subset O_X, x$.

**Proof:**

\[ s: \mathcal{F}_x^* \rightarrow O_X \]

\[ \xymatrix{ \mathcal{F}_x^* \ar[r] & O_X \ar[r] & i_x O_{Z(s)} \ar[r] & 0 } \]

0 $\rightarrow$ \( O_{Z(s)} \rightarrow O_X \rightarrow i_x O_{Z(s)} \rightarrow 0 \)

where \( i: Z(s) \rightarrow X \)

\[ x \in Z(s) \iff (i_x O_{Z(s)})_x \neq 0 \]

\[ \Rightarrow \quad O_{Z(s), x} \not\cong O_{X, x} \]

\[ \Rightarrow \quad O_{Z(s), x} \subset M_x \subset O_{X, x} \]

\[ \Rightarrow \quad s(\mathcal{F}_x^*) \subset M_x. \]

**Claim 2:** Suppose \( \mathcal{F}_x \) is locally free of finite rank \( r \).

Then, for \( x \in X, \mathcal{F}_x \) is a free \( O_X, x \)-module of rank \( r \).
\[ F_x \cong C \times_1 C \]

\[ \Rightarrow \quad F_x^* = \text{Hom}_{C \times_1 C} (F_x, C \times_1 C) \cong \text{Hom}_{C \times_1 C} (C \oplus C, C \times_1 C) \]

The condition \( s_x (F_x^*) \subset M_x \) means \( s (x) \in \mathfrak{m}_x F_x \).

Indeed, if we pass to the quotient by \( M_x \):

\[ F_x / M_x F_x \cong C \times_1 C / M_x \cong k (x) \oplus k (x) \]

and \( F_x^*/M_x F_x^* \cong (F_x / M_x F_x)^* \cong (k (x) \oplus k (x))^* \)

\( s_x (F_x^*) \subset M_x \) \( \Rightarrow \) \( s_x \) induces the linear map \( M_x (k (x) \oplus k (x))^* \)

\( \Rightarrow \) \( s_x = 0 \in k (x) \oplus k (x) = F_x / M_x F_x \)

\( \Rightarrow \) \( s (x) \in \mathfrak{m}_x F_x \).
Note: This fails if \( X \) is not locally free:

\[ \text{e.g.: } \mathcal{O} = \text{skyline sheaf supported on a proper closed subset of } X. \]

Then \( \mathcal{O}^* = \mathcal{O}_X \) the zero sheaf on \( X \)

Homogeneous ideals of subsets of projective space:

Assume \( S := R \langle X_0, \ldots, X_n \rangle \), \( R \) commutative

\( X := \text{Proj} \ S = \mathbb{P}^n_R \)

We saw that \( \Gamma(X, \mathcal{O}_X(d)) = S_d \quad \forall \ d \in \mathbb{Z} \)

i.e., \( \Gamma_*(\mathcal{O}_X) = S \).

**Def:** (1) For a subset \( Y \subset X \), define

\[ I_{Y,d} := \{ s \in S_d \mid Y \subset \omega Z(s) \} \]
the set of homogeneous polynomials of degree vanishing on \( Y := \{ s \in S^d \mid \forall x \in X, s(x) \in m_x \} \).

(2) For a closed subscheme \( Y \subset X \) with ideal sheaf \( \mathcal{I}_Y \), define
\[
I_{Y,d} := \{ s \in S^d \mid \mathcal{I}_Y^2(s) \subset \mathcal{I}_Y \} = \{ s \in S^d \mid Y \subset Z(s) \text{ as subschemes} \}.
\]

**Ex:** If \( Y \) is reduced, the two definitions are equal.

**Def:** The homogeneous ideal of a subset or closed subscheme \( Y \subset X = \mathbb{P}^n_R \) is
\[
I_Y := \bigoplus_{d \in \mathbb{Z}} I_{Y,d} \subset S
\]
The homogeneous coordinate ring of \( Y \) is \( S(Y) := \frac{S}{I(Y)} \).
Remark: Choose \( s \in \mathcal{S}_d \) and suppose \( Y \subset X \) is a closed subscheme. We have \( s \in \mathcal{I}_Y \) if \( I_z(s) \subset \mathcal{I}_Y \).

\( I_z(s) \) is the image of \( s : \mathcal{O}_X(d)^* \to \mathcal{O}_X \)

Recall that \( \mathcal{O}_X(d)^* \cong \mathcal{O}_X(-d) \), so

\( I_z(s) \) is the image of \( s : \mathcal{O}_X(-d) \to \mathcal{O}_X \)

\( I_z(s) \subset \mathcal{I}_Y \) \( \iff \) \( \mathcal{O}_X(-d) \to \mathcal{O}_X \to \mathcal{O}_Y \) factor

Twist by \( \mathcal{O}_X(d) : \mathcal{O}_X(-d) \otimes \mathcal{O}_X(d) \to \mathcal{O}_X(d) \to \mathcal{O}_Y(d) \)
Fact: \[ C_X \xrightarrow{S} C_X(d) \] is "multiplication" by \( S \):

\[ \forall u \in C_X \quad C_X(u) \xrightarrow{S} C_X(d)(u) \]

\[ \exists \phi | s | u \in C_X(d)(u) \]

\( S \) factors through \( I_y(d) \) \( \iff \) \( s \in \Gamma (X, I_y(d)) \)

More generally: \( \forall \) \( C_X \)-module \( \mathcal{F} \),

\[ \Gamma (X, \mathcal{F}) = \text{Hom}_{C_X} (C_X, \mathcal{F}) \]

\[ s \mapsto \left( f \mapsto \sum s | u \text{ in any } u \right) \]

\[ \varphi (1) \leftarrow \varphi \]

\( \Rightarrow \) \( s \in \Gamma (X, I_y(d)) \) \( \iff \) \( s : C_X \rightarrow I_y(d) \)

Conclusion: \( s \in I_{Y, d} \) \( \iff \) \( s \in \Gamma (X, I_y(d)) \).
\[ I_{y,d} = \Gamma(X, I_y(d)) \]

Note that \( I_y \subset O_X \Rightarrow I_y(d) \subset O_X(d) \]
\[ \Rightarrow \Gamma(I_y(d)) \subset \Gamma(O_X(d)) \subset I_{y,d} \]

\[ \Rightarrow I_y = \bigoplus_{d \in \mathbb{Z}} I_{y,d} = \bigoplus_{d \in \mathbb{Z}} \Gamma(I_y(d)) \]

\[ =: \Gamma_{\mathbb{Z}}(I_y) \]

Recall: If \( S \) is finitely generated by \( S_1 \) as an \( S_0 \)-alg. and \( T \) is quasi-coherent, then \( T \times (\tilde{S}_1) \tilde{\cong} T_0 \)

This is true for \( S = R[X_0, \ldots, X_n] \) and \( T = I_y \), so \( \Gamma_{\mathbb{Z}}(I_y) \tilde{\cong} I_y \), i.e., \( I_y \tilde{\cong} I_y \).
Main Lemma: There is a natural isomorphism

\[ g : Y \cong \text{Proj} \cdot S(Y) \]

Proof: The quotient morphism \( q : S \rightarrow S(Y) = S/I(Y) \)

induces a morphism \( f : \text{Proj} \cdot S(Y) \rightarrow \text{Proj} \cdot S \).

(\( f \) is defined everywhere because \( q \) is surjective)

We will show that \( f \) factors through an isomorphism.

\[ \text{Proj} \cdot S(Y) \cong Y \hookrightarrow X := \text{Proj} \cdot S \]

We have the exact sequence

\[ 0 \rightarrow I_Y \rightarrow S \rightarrow S(Y) \rightarrow 0 \quad \text{(by def.)} \]

Lemma: The \( \sim \) functor is exact.
Proof: Suppose given an exact sequence of graded $S$-modules:

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

Then, for $t \in S_d$, we have the exact sequences:

$$0 \rightarrow M',[t-1] \rightarrow M,[t-1] \rightarrow M'',[t-1] \rightarrow 0$$

and

$$0 \rightarrow M',[t-1],_0 \rightarrow M,[t-1],_0 \rightarrow M'',[t-1],_0 \rightarrow 0$$

on the affine scheme $U := Spec S[t^{-1}],_0 \subset X$.

$$\Rightarrow 0 \rightarrow \widetilde{M}',(t),_0 \rightarrow \widetilde{M},(t),_0 \rightarrow \widetilde{M}'',(t),_0 \rightarrow 0$$

on $U,(t)_0$.

$$\Rightarrow 0 \rightarrow \tilde{M}',(t) \rightarrow \tilde{M},(t) \rightarrow \tilde{M}'',(t) \rightarrow 0$$

The open sets $U,(t)$ cover $X \Rightarrow 0 \rightarrow \tilde{M}',\tilde{M} \rightarrow \tilde{M}'', \rightarrow 0$ is exact on $X$. $\Box$