More generally: \( R = k \) alg. closed field
\[
m = 1, \text{ any } d, \quad P^1 \hookrightarrow P^m = P^d, \quad m = (1 + d) - 1 = d
\]
Def: The image is "the" rational normal curve of degree \( d \). It is always the zero locus of some quadratic polynomials.

Example 2: Segre embeddings:
\[
\begin{align*}
P^l \times P^m & \hookrightarrow P^n \\
\text{where } l, m & \text{ are arbitrary, } n = (l+1)(m+1) - 1.
\end{align*}
\]
In homogeneous coordinates (when \( R = k \))
\[
((a_0, \ldots, a_l), (b_0, \ldots, b_m)) \mapsto (a_0 b_0, a_0 b_1, \ldots, a_l b_m)
\]
To define the morphism on the Proj:
\[
\Gamma := R[Z_0, \ldots, Z_n]
\]
\[ S := R[X_0, \ldots, X_l] \times R[Y_0, \ldots, Y_m] \]

where, for two graded \( R \)-algebras \( R_1 \) and \( R_2 \), the Cartesian \( R \)-product \( R_1 \times R_2 \) is, by definition,

\[ R_1 \times R_2 := \bigoplus_{d \geq 0} R_1, d \otimes R_2, d \]

(Ex. II.5.11): There is a natural isomorphism

\[ \text{Proj} \, R_1 \times \text{Proj} \, R_2 \cong \text{Proj} \left( R_1 \times R_2 \right) \]

\text{Spec} R \quad \text{Spec} R

\[ \text{s.t.} \quad f^* \circ \text{Proj} \, R_1 (1) \otimes g^* \circ \text{Proj} \, R_2 (1) \cong \circ \text{Proj} (R_1 \times R_2) (1) \]

Apply this to \( R_1 = R[X_0, \ldots, X_l] \), \( R_2 = R[Y_0, \ldots, Y_m] \)

then \( S = R_1 \times R_2 \)
The elements \( X_i \otimes Y_j \in R_1 \times R_2 = S \) form a set of generators for \( S \) as a \( R \)-algebra.

Choosing an ordering of \( \{ X_i \otimes Y_j \} \) means giving a bijection

\[
\{ X_i \otimes Y_j \} \leftrightarrow \{ Z_0, \ldots, Z_n \}
\]

which defines a surjective homomorphism of \( R \)-algebras

\[
T = R[Z_0, \ldots, Z_n] \longrightarrow S = R_1 \times R_2
\]

\[
Z_i \longmapsto X_i \otimes Y_j
\]

This defines a closed embedding \( \mathbb{P}^l \times \mathbb{P}^m \hookrightarrow \mathbb{P}^n \).

\textbf{Special case:} \quad \text{The smooth quadric in} \quad \mathbb{P}^3 : (R = \text{k alg. closed field})

\[
l = m = 1 \quad \Rightarrow \quad n = (l+1)(m+1) - 1 = 3
\]

\[
\mathfrak{f} : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3
\]

\[
\begin{align*}
(\alpha_0, \alpha_1) \; & \; , \; (\beta_0, \beta_1) \longmapsto (a_0 \cdot b_1, a_0 \cdot b_1, a_1 \cdot \beta_0, a_1 \cdot \beta_1) = (c_0, c_1, c_2, c_3)
\end{align*}
\]
relation: \[ c_0 c_3 - c_1 c_2 \]
\[ \Rightarrow f(P^1 \times P^1) \subset \mathbb{Z} (Z_0 Z_2 - Z_1 Z_2) \] quadratic

can show =

Understanding the better in the projective case:

Recall that in the affine case: \( \Gamma(\text{Spec} \mathcal{A}, \mathcal{M}) = M \)

and we can recover \( \mathcal{M} \) from \( M \).

In the projective case: \( \Gamma(\mathbb{P}^n, O(d)) = S_d = S[d] \) when \( S = \mathbb{R}[X_0, \ldots, X_n] \). \( S[d] \)

so we only recover the degree 0 piece and we cannot recover the sheaf \( O(d) = S[d] \).

Given a quasi-coherent sheaf \( \mathcal{F}_0 \), it would be nice if we could find \( M \) s.t. \( \mathcal{M} \cong \mathcal{F}_0 \).
Definition: For a sheaf $\mathcal{F}$ on $\text{Proj}^* S$ ($S$ any graded ring), put $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}_{\text{Proj}^* S}(n)$ the "twist" of $\mathcal{F}$ by $n$.

Define $\Gamma_* (\mathcal{F}) := \bigoplus_{n \in \mathbb{Z}} \Gamma (\text{Proj}^* S, \mathcal{F}(n))$

this is a graded module over $S$.

Remark: There is a natural morphism $\varphi : \Gamma_* (\mathcal{F}) \to \mathcal{F}$ defined on basic open sets as follows: $f \in S_m, m > 0$

$U_f \subset \text{Proj}^* S$

$\Gamma (U_f, \Gamma_* (\mathcal{F})) = \Gamma_* (\mathcal{F})[pf^{-1}] \to \mathcal{F}(U_f)$

given $s \in \Gamma_* (\mathcal{F})[pf^{-1}]$, $\exists t \in \Gamma_* (\mathcal{F})$ and $d > 0$

s.t. $s = \frac{t}{pf^d}$ and $\deg t = dm$, i.e., $t \in \Gamma (\mathcal{F}(dm))$
we define the image of $s$ in $\Gamma(U_f, F_t)$ is, by def., the image of $t \otimes f^{-d}$ via the tensor product map

\[ \Gamma(U_f, \mathcal{F}(d\omega)) \otimes \Gamma(U_f, G(-d\omega)) \rightarrow \Gamma(U_f, \mathcal{F}(d\omega)(-d\omega)) \]

\[ t|_{U_f} \otimes (f|_{U_f})^{-d} \rightarrow \varphi(s) \in \Gamma(U_f, \mathcal{F}) \]

**Prop. II.5.15:** When $S$ is finitely generated by $S$, as an $S_o$-algebra and $F_t$ is quasi-coherent, the morphism $\varphi$ is an isomorphism.
Some important definitions:

**Def 1:** For any scheme \( X \), \( \mathbb{P}_X^n := \mathbb{P}^n_Z \times \text{Spec} \mathbb{Z} \to X \).

(\( X \to \text{Spec} \mathbb{Z} \), obtained from \( \mathbb{Z} \to \Gamma(X, G_X) \).

The twisting sheaf \( G_{\mathbb{P}^n_X}^1 \) is, by def.,

\[
G_{\mathbb{P}^n_X}^1 := \mathfrak{p}_1^* G_{\mathbb{P}^n_Z}^1
\]

**Def 2:** A morphism of schemes \( W \to X \) is an embedding if it is a composition \( W \to Z \to X \) where \( W \to Z \) is an open embedding and \( Z \to X \) is a closed embedding.
Def. 3: For any morphism of schemes $\pi: X \to Y$, an invertible sheaf $L$ on $X$ is very ample relative to $\pi$ if there exists an (ample) embedding $i: X \hookrightarrow \mathbb{P}^n_Y$ such that

$$L = i^* O_{\mathbb{P}^n_Y}(1)$$

Def. 4: Given a sheaf $F$ and a collection $\{s_i, i \in I\}$ of global sections of $F$ on $X$, we say $F$ is generated by $\{s_i, i \in I\}$ if, for all $x \in X$, the stalk $F_x$ is generated as an $O_{X,x}$-module by the genus $\{s_i(x), i \in I\}$. We say $F$ is generated by global sections, or, globally generated, if it is generated by some collection of global sections.