More generally: $R = k$ alg closed field

$n=1$, <u>any d</u>, $\mathbb{P}^1 \hookrightarrow \mathbb{P}^m = \mathbb{P}^d$ $\quad m = \binom{1+d}{d} - 1 = d$

Def: The image is "the" rational normal curve of degree d. It is always the zero locus of some quadratic polynomials.

Example 2: **Segre embeddings:** $\mathbb{P}^l \times_R \mathbb{P}^m \hookrightarrow \mathbb{P}^n$

where l, m are arbitrary, $n = (l+1)(m+1) - 1$.

In homogeneous coordinates (when $R = k$)

$$((a_0, \dots, a_l), (b_0, \dots, b_m)) \longmapsto (a_0 b_0, a_0 b_1, \dots, a_l b_m) = (c_0, \dots, c_n)$$

To define the morphism on the Proj:

$$T := R[Z_0, \dots, Z_n]$$

$$S := R[X_0, \cdots, X_\ell] \times_R R[Y_0, \cdots, Y_m]$$

where, for two graded R-algebras R_1 and R_2, the Cartesian R-product $R_1 \times_R R_2$ is, by definition,

$$R_1 \times_R R_2 := \bigoplus_{d \geq 0} R_{1,d} \otimes R_{2,d}$$

(Ex. II.5.11): There is a natural isomorphism

$$\mathrm{Proj}\, R_1 \times_{\mathrm{Spec}\, R} \mathrm{Proj}\, R_2 \cong \mathrm{Proj}(R_1 \times_R R_2)$$

s.t. $\quad p_1^* \mathcal{O}_{\mathrm{Proj}\, R_1}(1) \otimes_{\mathcal{O}} p_2^* \mathcal{O}_{\mathrm{Proj}\, R_2}(1) \cong \mathcal{O}_{\mathrm{Proj}(R_1 \times_R R_2)}(1)$

Apply this to $R_1 = R[X_0, \dots, X_\ell]$, $R_2 = R[Y_0, \dots, Y_m]$

then $S = R_1 \times_R R_2$

The elements $X_i \otimes Y_j \in R_1 \times_R R_2 = S$ form a set of generators for S as an R-algebra.

Choosing an ordering of $\{X_i \otimes Y_j\}$ means giving a bijection

$$\{X_i \otimes Y_j\} \longleftrightarrow \{Z_0, \dots, Z_n\}$$

which defines a surjective homomorphism of R-algebras

$$T = R[Z_0, \dots, Z_n] \twoheadrightarrow S = R_1 \times_R R_2$$

$$Z_p \longmapsto X_i \otimes Y_j$$

This defines a closed embedding $\mathbb{P}^l \times_{\operatorname{Spec} R} \mathbb{P}^m \hookrightarrow \mathbb{P}^n$.

<u>First case</u>: The smooth quadric in $\mathbb{P}^3$: ($R = k$ alg. closed field)

$$l = m = 1 \qquad n = (l+1)(m+1) - 1 = 3$$

$$f: \mathbb{P}^1 \times_{\operatorname{Spec} k} \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$$

$$((a_0, a_1), (b_0, b_1)) \longmapsto (a_0 b_0, a_0 b_1, a_1 b_0, a_1 b_1) = (c_0, c_1, c_2, c_3)$$

relation: $c_0 c_3 - c_1 c_2$

$$\Rightarrow f(\mathbb{P}^1 \times \mathbb{P}^1) \subset Z(Z_0 Z_3 - Z_1 Z_2) \text{ quadric}$$

can show $=$

Understanding the $\sim$ better in the projective case:

Recall that in the affine case: $\Gamma(\operatorname{Spec} A, \tilde{M}) = M$ and we can recover $\tilde{M}$ from M.

In the projective case: $\Gamma(\mathbb{P}^n_R, \mathcal{O}(d)) = S_d = S[d]_0$
where $S = R[X_0, \ldots, X_n]$ $\qquad \underbrace{S_d}_{= S[d]}$

so we only recover the degree 0 piece and we cannot recover the sheaf $\mathcal{O}(d) = \widetilde{S[d]}$.

Given a quasi-coherent sheaf $\mathcal{F}$, it would be nice if we could find M s.t. $\tilde{M} \cong \mathcal{F}$.

<u>Definition:</u> For a sheaf $\mathcal{F}$ on $\operatorname{Proj} S$ (S any graded ring),

put $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}_{\operatorname{Proj} S}(n)$ the "twist" of $\mathcal{F}$ by n.

Define $\Gamma_*(\mathcal{F}) := \bigoplus_{n \in \mathbb{Z}} \Gamma(\operatorname{Proj} S, \mathcal{F}(n))$

this is a graded module over S.

<u>Remark:</u> There is a natural morphism

$$\varphi : \widetilde{\Gamma_*(\mathcal{F})} \longrightarrow \mathcal{F}$$

defined on basic open sets as follows: $f \in S_m$, $m > 0$

$U_f \subset \operatorname{Proj} S$

$$\Gamma(U_f, \widetilde{\Gamma_*(\mathcal{F})}) = \Gamma_*(\mathcal{F})[f^{-1}]_0 \overset{?}{\dashrightarrow} \mathcal{F}(U_f)$$

given $s \in \Gamma_*(\mathcal{F})[f^{-1}]_0$, $\exists$ $t \in \Gamma_*(\mathcal{F})$ and $d > 0$

s.t. $s = \frac{t}{f^d}$ and $\deg t = dm$, i.e., $t \in \Gamma(\mathcal{F}(dm))$

we define the image of s in $\Gamma(U_f, \mathcal{F})$ is, by def., the image of $t \otimes f^{-d}$ via the tensor product map

sections of presheaf to sections of associated sheaf

$$\Gamma(U_f, \mathcal{F}(dm)) \otimes \Gamma(U_f, \mathcal{O}(-dm)) \longrightarrow \Gamma(U_f, \mathcal{F}(dm)(-dm))$$

$$\Vert$$

$$t|_{U_f} \otimes (f|_{U_f})^{-d} \longmapsto \varphi(s) \in \Gamma(U_f, \mathcal{F})$$

<u>Prop. II.5.15</u>: When S is finitely generated by S_1 as an S_0-algebra and $\mathcal{F}$ is quasi-coherent, the morphism φ is an isomorphism.

Some important definitions:

Def 1: For any scheme X, $\mathbb{P}^n_X := \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} X$

$\left(X \longrightarrow \operatorname{Spec}\mathbb{Z} \text{, obtained from } \mathbb{Z} \longrightarrow \Gamma(X, \mathcal{O}_X) \right)$,

The twisting sheaf $\mathcal{O}_{\mathbb{P}^n_X}(1)$ is, by def.,

$$\mathcal{O}_{\mathbb{P}^n_X}(1) := p_1^* \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)$$

Def 2: A morphism of schemes $W \longrightarrow X$ is an embedding if it is a composition $W \hookrightarrow Z \hookrightarrow X$ where $W \hookrightarrow Z$ is an open embedding and $Z \hookrightarrow X$ is a closed embedding.

Def. 3: For any morphism of schemes $\pi : X \to Y$, an invertible sheaf $\mathcal{L}$ on X is very ample relative to π if $\exists$ an (π-)embedding $i : X \hookrightarrow \mathbb{P}^n_Y$ s.t.

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "\pi"'] & \mathbb{P}^n_Y \arrow[d] \\ & Y \end{tikzcd}$$

$$\mathcal{L} \cong i^* \mathcal{O}_{\mathbb{P}^n_Y}(1)$$

Def. 4: Given a sheaf $\mathcal{F}$ and a collection $\{s_i, i \in I\}$ of global sections of $\mathcal{F}$ on X, we say $\mathcal{F}$ is generated by $\{s_i, i \in I\}$ if, $\forall x \in X$, the stalk $\mathcal{F}_x$ is generated as an $\mathcal{O}_{X,x}$-module by the germs $\{s_i(x), i \in I\}$.

We say $\mathcal{F}$ is generated by global sections, or, globally generated, if it is generated by some collection of global sections.