<u>Proof of</u> the theorem from the last class about quasi-coherent sheaves:

Choose an open affine set $U = \operatorname{Spec} A \subset X$.

As in the proof of the previous lemma, we can cover U with basic open sets $\operatorname{Spec} A[g_i^{-1}]$ s.t. $\forall i$ $\exists$ $A[g_i^{-1}]$-mod M_i

$$\mathcal{F}\big|_{\operatorname{Spec} A[g_i^{-1}]} \cong \widetilde{M}_i$$

Put $M := \Gamma(U, \mathcal{F})$. We show $\mathcal{F} \cong \widetilde{M}$.

Homework (Ex. II.5.3) $\exists$ morphism of sheaves

$$\alpha : \widetilde{M} \longrightarrow \mathcal{F}$$

we show α is an isomorphism.

We know $\mathcal{F}\big|_{D(g_i)} \cong \widetilde{M}_i$ and $M_i = \Gamma(D(g_i), \mathcal{F})$.

The previous lemma implies $M_i = M[g_i^{-1}]$

(induced by restriction $M \longrightarrow M_i$
$$\underset{\Gamma(V, \mathcal{O}_X)}{\overset{\shortparallel}{M}} \longrightarrow \underset{\Gamma(D(g_i), \mathcal{O}_X)}{\overset{\shortparallel}{M_i}}$$
show that it factors through $M \to M[g_i^{-1}]$)

So we have
$$\mathcal{O}_X\big|_{D(g_i)} \cong \tilde{M_i} \cong \widetilde{M[g_i^{-1}]} \cong \tilde{M}\big|_{D(g_i)}$$
$$\xleftarrow[\alpha|_{D(g_i)}]{}$$

$\Rightarrow$ $\alpha|_{D(g_i)}$ is an isom. $\forall i$ $\Rightarrow$ α is an isom. $\square$

A few words about sheaves of ideals.

<u>Def</u>: A sheaf of ideals on X is an $\mathcal{O}_X$-submodule of $\mathcal{O}_X$.

For any closed subscheme Z of X, we define the sheaf of ideals $\mathcal{I}_Z$ associated to Z of elements of $\mathcal{O}_X$ vanishing on Z. Formally, let $i: Z \hookrightarrow X$ be the inclusion morphism, we have $i^\#: \mathcal{O}_X \twoheadrightarrow i_*\mathcal{O}_Z$.

$$\mathcal{I}_Z := \ker\left(i^\#: \mathcal{O}_X \twoheadrightarrow i_*\mathcal{O}_Z\right).$$

$$\Rightarrow \quad 0 \to \mathcal{I}_Z \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0 \text{ is exact.}$$

$i_*\mathcal{O}_Z$ is a pushforward of a coherent sheaf $\Rightarrow$ quasi-coherent. $\mathcal{I}_Z$ is the kernel of a morphism of quasi coherent sheaves $\Rightarrow$ $\mathcal{I}_Z$ is quasi-coherent.

X noetherian $\Rightarrow$ $\mathcal{I}_Z$ is coherent

On any open affine $U = \operatorname{Spec} A \subset X$, by the theorem, $\exists$ an ideal $I \subset A$ s.t. $\mathcal{I}_Z\big|_U \cong \tilde{I}$ and Z is defined by the ideal I. In particular (exercise), we have $Z \cong \operatorname{Spec} A/I$.

In particular, if $X = \operatorname{Spec} A$ is affine, we have a 1-to-1 correspondence between closed subschemes of X and ideals $I \subset A$.

If X is not affine, we have a 1-to-1 correspondence between closed subschemes of X and coherent sheaves of ideals.

Note: $i_* \mathcal{O}_Z$ is quasi-coherent and a quotient of $\mathcal{O}_X$

$\Rightarrow$ on open affine sets $i_* \mathcal{O}_Z \big|_{\operatorname{Spec} A} \cong \tilde{M}$

$\mathcal{O}_X \big|_{\operatorname{Spec} A} = \mathcal{O}_{\operatorname{Spec} A} = \tilde{A}$

$$\mathcal{O}_X \twoheadrightarrow i_* \mathcal{O}_Z$$

$$\Rightarrow \quad \tilde{A} \twoheadrightarrow \tilde{M}$$

$\Rightarrow$ M is finitely generated (see future homework)

So $i_* \mathcal{O}_Z$ is <u>coherent</u>.

In general, any quotient of a coherent sheaf is coherent.

Some fact about Proj :

Let $S = \bigoplus_{d=0}^{\infty} S_d$ be a graded ring and put

$S_+ := \bigoplus_{d=1}^{\infty} S_d$ if S_0 is a field, this is the largest homogeneous ideal.

<u>recall</u>: $\operatorname{Proj} S = \{ \mathfrak{p} \mid \mathfrak{p} \not\supset S_+ \text{ homogeneous} \}$

closed subsets $Z(I) = \{ \mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{p} \supset I \}$

$I \subset S$ homogeneous ideal.

basic open sets, for $f \in S_d$ $d > 0$

$$U_f := \{ \mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p} \} = \operatorname{Proj} S \setminus Z(f)$$

$S_{(f)} := S[f^{-1}]_0$ the ring of degree 0 elements of $S[f^{-1}]$.

We saw $\quad U_f \cong \operatorname{Spec} S_{(f)} = \operatorname{Spec} S[f^{-1}]_0$

$$\mathcal{O}_{\operatorname{Proj} S}(U_f) = S_{(f)} = S[f^{-1}]_0$$

The requirement $\mathfrak{p} \not\supset S_+$ ensures that the basic open sets U_f cover $\operatorname{Proj} S$, in fact they form a basis of the topology of $\operatorname{Proj} S$.

For any graded S-module $M = \bigoplus_{d \in \mathbb{Z}} M_d$, we define the sheaf $\tilde{M}$ on $\operatorname{Proj} S$ via

$$\tilde{M}(U_f) := M[f^{-1}]_0 =: M_{(f)}$$

$$\tilde{M}|_{U_f} \cong \widetilde{M[f^{-1}]_0}$$ similar to $U_f \cong \operatorname{Spec} S_{(f)}$
(unwind the definitions)

$\Rightarrow \quad \tilde{M}$ is quasi-coherent.

We generalize the sheaves $\mathcal{O}(n)$ to $\operatorname{Proj} S$:

Put $\quad S[n] := \bigoplus_{d \in \mathbb{Z}} S[n]_d \quad$ where $\quad S[n]_d := S_{n+d}$

the shift of S by n, for $n \in \mathbb{Z}$.

Define $\quad \mathcal{O}_{\operatorname{Proj} S}(n) := \widetilde{S[n]}$

Ex: When $S = R[X_0, \dots, X_n]$ for a commutative ring R, this definition agrees with the previous definition of $\mathcal{O}(n)$. In other words, $\mathcal{O}(n)$ is free on $U_i := U_{X_i}$ and the transition functions are $\left(\frac{X_j}{X_i}\right)^n$.

Some examples: (Ex II.2.14):

Let $\varphi : S \to T$ be a hom. of graded rings ($\varphi(S_d) \subset T_d$).

Put $U := \{ \mathfrak{p} \in \operatorname{Proj} T \mid \mathfrak{p} \not\supset \varphi(S_+) \}$.

Then U is an open subset of $\operatorname{Proj} T$ and φ defines a natural morphism of schemes $f : U \to \operatorname{Proj} S$.

At the level of sets, $f(\mathfrak{p}) := \varphi^{-1}(\mathfrak{p})$.

One checks that f is a continuous map of top. spaces, in fact $f^{-1}(U_g) = U_{\varphi(g)}$ $\forall\, g \in S_+$ homogeneous.

At the level of sheaves, φ induces a graded morphism

$$S[g^{-1}] \longrightarrow T[\varphi(g)^{-1}]$$

$$\Rightarrow \quad S[g^{-1}]_0 \longrightarrow T[\varphi(g)^{-1}]_0$$

If $\exists\ d>0$ s.t. φ induces isomorphisms

$$\varphi_n : S_n \xrightarrow{\cong} T_n \quad \forall\, n \geq d\ , \text{ then}$$

$$U = \operatorname{Proj} T \quad \text{and} \quad f : \operatorname{Proj} T \xrightarrow{\cong} \operatorname{Proj} S\ .$$

For this, use the fact that $\forall\, g \in S_+$ homogeneous,

and, $\forall\ n>0$, $\quad U_g = U_{g^n}\ ,\ S[g^{-1}] \cong S[(g^n)^{-1}]$

and $S[g^{-1}]_0 = S[(g^n)^{-1}]_0$.

<u>Example 1:</u> The d-uple embedding: R a comm. ring

$$\mathbb{P}^n := \mathbb{P}^n_R \qquad m := \binom{n+d}{d} - 1$$

<u>counting exercise:</u> $T := R[Y_0, \dots, Y_n]$, the number of degree d monomials in $Y_0, \dots, Y_n$ is $\binom{n+d}{d}$

Put $S := R[X_0, \dots, X_m]$

Define a morphism of graded R-algebras $\varphi : S \twoheadrightarrow T$ by first choosing an ordering of all the monomials of degree d in $Y_0, \dots, Y_n$ and sending X_i to the i-th monomial. Using ex. II.2.14, we obtain a morphism

$$\begin{array}{ccc} f : \operatorname{Proj} T & \longrightarrow & \operatorname{Proj} S \\ \| & & \| \\ \mathbb{P}^n & \hookrightarrow & \mathbb{P}^m \end{array}$$

can show closed embedding

<u>Def</u>: this is the d-uple embedding.

A little more concretely, in terms of homogeneous coordinates, f sends $(b_0,\dots,b_n)\in\mathbb{P}^n$ to the point of coordinates $(a_0,\dots,a_m)\in\mathbb{P}^m$ where a_i is the i-th monomial of degree d in $b_0,\dots,b_n$.

First cases: $R=k$ alg. closed field

(1) $n=1$, $d=2$ $\qquad m=\binom{1+2}{1}-1=2$

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^2 \quad \text{image is a conic}$$

$$(b_0,b_1)\longmapsto(a_0,a_1,a_2)=(b_0^2,b_0b_1,b_1^2)$$

relation: $a_0a_2-a_1^2=0$

$\Rightarrow$ image $\subset Z(X_0X_2-X_1^2)$

can show $=$.

(2) <u>Twisted cubics</u>: $n=1$, $d=3$, $m=\binom{1+3}{3}-1=3$

$$f: \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$$

$$(t_0,t_1) \longmapsto (a_0,a_1,a_2,a_3)=(t_0^3, t_0^2t_1, t_0t_1^2, t_1^3)$$

relations: $a_0a_3-a_1a_2$, $a_0a_2-a_1^2$, $a_1a_3-a_2^2$

$$f(\mathbb{P}^1) \subset Z(X_0X_3-X_1X_2, X_0X_2-X_1^2, X_1X_3-X_2^2)$$

can show $=$.