Proof of the theorem from the last class about quasi-coherent sheaves:

Choose an open affine set \( U = \text{Spec} \, A \subset X \).

As in the proof of the previous lemma, we can cover \( U \) with basic open sets \( \text{Spec} \, A[\mathfrak{g}_i] \) s.t. \( \forall \, i \in I \) on \( A[\mathfrak{g}_i] \) and \( M_i \):

\[
\text{Spec} \, A[\mathfrak{g}_i] \supseteq M_i
\]

Put \( M := \Gamma(U, \mathcal{F}) \). We show \( \mathcal{F} \cong M \).

Homework (Ex. II.5.3) \( \exists \) morphism of sheaves

\[
\alpha : M \to \mathcal{F}
\]

we show \( \alpha \) is an isomorphism.

We know \( \mathcal{F} \mid_{D(\mathfrak{g}_i)} \cong M_i \) and \( M_i = \Gamma(D(\mathfrak{g}_i), \mathcal{F}) \).
The previous lemma implies \( M_i = M[g_i^-] \).

(Induced by restriction \( M \to M_i \))

\[ \Gamma(U, \mathscr{F}) \cong \Gamma(D(g_i), \mathscr{F}) \]

show that it factors through \( M \to M[g_i^-] \).

So we have \( \mathfrak{F} \)\( \xrightarrow{D(g_i)} \)\( \cong \tilde{M}_i \cong M[g_i^-] \cong \tilde{M} \)\( \xrightarrow{D(g_i)} \).

\( \Rightarrow \) \( \alpha \) is an isom. \( \forall i \Rightarrow \alpha \) is an isom. \( \square \)
A few words about sheaves of ideals.

Def: A sheaf of ideals on \( X \) is an \( O_X \)-submodule of \( O_X \).

For any closed subscheme \( Z \) of \( X \), we define the sheaf of ideals \( \mathcal{I}_Z \) associated to \( Z \) of elements of \( O_X \) vanishing on \( Z \). Formally, let \( i: Z \to X \) be the inclusion morphism, we have \( i^\#: O_X \to i_* O_Z \).

\[
\mathcal{I}_Z := \ker (i^\#: O_X \to i_* O_Z).
\]

\[
\Rightarrow \quad O \to \mathcal{I}_Z \to O_X \to i_* O_Z \to 0 \quad \text{is exact.}
\]

\( i_* O_Z \) is a pushforward of a coherent sheaf \( \Rightarrow \) quasi-coherent. \( \mathcal{I}_Z \) is the kernel of a morphism of
quasi-coherent sheaves \( \Rightarrow \mathcal{F}_Z \) is quasi-coherent.

\( X \) noetherian \( \Rightarrow \mathcal{F}_Z \) is coherent.

On any open affine \( U = \text{Spec } \mathcal{A} \subset X \), by the theorem, \( \exists \) an ideal \( I \subset \mathcal{A} \) s.t. \( \mathcal{F}_Z|_U \cong \mathcal{I} \) and \( Z \) is defined by the ideal \( I \). In particular (exercise), we have \( Z \cong \text{Spec } \mathcal{A}/I \).

In particular, if \( X = \text{Spec } \mathcal{A} \) is affine, we have a 1-to-1 correspondence between closed subschemes of \( X \) and ideals \( I \subset \mathcal{A} \).

If \( X \) is not affine, we have a 1-to-1 correspondence between closed subschemes of \( X \) and coherent sheaves of ideals.
Note: \( i_* G_2 \) is quasi-coherent and a quotient of \( O_X \)

\[
\Rightarrow \text{ on open affine set } \quad i_* G_2 \mid_{\text{spec } A} \cong \tilde{M}
\]

\[
G_X \mid_{\text{spec } A} = \text{ spec } A = \tilde{A}
\]

\[
G_X \rightarrow i_* G_2
\]

\[
\Rightarrow \tilde{A} \rightarrow \tilde{M}
\]

\[
\Rightarrow \text{ } M \text{ is finitely generated (see future homework)}
\]

So \( i_* G_2 \) is coherent.

In general, any quotient of a coherent sheaf is coherent.
Some fact about $\text{Proj}$:

Let $S = \bigoplus_{d=0}^{\infty} S_d$ be a graded ring and put

$$S_+ := \bigoplus_{d=1}^{\infty} S_d$$

If $S_0$ is a field, this is the largest homogeneous ideal.

Recall: $\text{Proj } S = \{ \mathfrak{p} \mid \mathfrak{p} \not\in S_+ \text{ homogeneous} \}$

closed subset $\mathcal{Z}(I) = \{ \mathfrak{p} \in \text{Proj } S \mid \mathfrak{p} \supset I \}$
$I \subset S$ homogeneous ideal.

basic open set, for $f \in S_d$, $d > 0$

$$U_f := \{ \mathfrak{p} \in \text{Proj } S \mid f \not\in \mathfrak{p} \} = \text{Proj } S \setminus \mathcal{Z}(f)$$

$S(f) := S[f^{-1}]$, the ring of degree 0 elements of $S[f^{-1}]$. 
We saw \( \mathcal{U}_f = \text{Spec } S(f) = \text{Spec } S[f^{-1}] \).
\[
\mathcal{O}_{\text{Proj}_S}(U_f) = S(f) = S[f^{-1}].
\]
The requirement \( f \not\in S^+ \) ensures that the basic open sets \( U_f \) cover \( \text{Proj}_S \), in fact they form a basis of the topology of \( \text{Proj}_S \).

For any graded \( S \)-module \( M = \bigoplus_{d \in \mathbb{Z}} M_d \), we define the sheaf \( \tilde{M} \) on \( \text{Proj}_S \) via the sheaf
\[
\tilde{M}(U_f) := M[f^{-1}] =: M(f)
\]
\[
\tilde{M}|_{U_f} \cong M[f^{-1}].
\]
(similar to \( U_f \cong \text{Spec } S(f) \)) (unwind the definitions)
$\Rightarrow \tilde{\mathcal{M}}$ is quasi-coherent.

We generalize the sheaves $\mathcal{O}(u)$ to $\text{Proj} S$:

Put $S[n] := \bigoplus S[n]_d$ where $S[n]_d := S_{n+d}$, where $\mathbf{deZ}$ denotes the shift of $S$ by $n$, for $n \in \mathbb{Z}$.

Define $\mathcal{O}_{\text{Proj} S}(n) := \tilde{S[n]}$.

Ex: When $S = R[X_0, \ldots, X_n]$ for a commutative ring $R$, this definition agrees with the previous definition of $\mathcal{O}(u)$.

In other words, $\mathcal{O}(u)$ is free on $U_i := U X_i$, and the transition functions are \((\frac{X_j}{X_i})^u\).
Some examples: (Ex III.2.14):

Let \( \varphi : S \to T \) be a hom. of graded rings \((\varphi(S_d) \subseteq T_d)\).

Put \( U := \{ \mathfrak{p} \in \text{Proj} \; T \mid \mathfrak{p} \neq \varphi(S_+) \} \).

Then \( U \) is an open subset of \( \text{Proj} \; T \) and \( \varphi \) defines a natural morphism of schemes \( f : U \to \text{Proj} \; S \).

At the level of sets, \( f(\mathfrak{p}) := \varphi^{-1}(\mathfrak{p}) \).

One checks that \( f \) is a continuous map of top. spaces, in fact \( f^{-1}(U g) = U \varphi(g) \), \( \forall \; g \in S_+ \) homogeneous.

At the level of sheaves, \( \varphi \) induces a graded morphism

\[
S \left[ g^{-1} \right] \to T \left[ \varphi(g)^{-1} \right]
\]

\[
\Rightarrow \; \; S \left[ g^{-1} \right]_0 \to T \left[ \varphi(g)^{-1} \right]_0.
\]
If $d > 0$ s.t. $\varphi$ induces isomorphisms

\[ \varphi_n : S_n \cong T_n \quad \forall \ n \geq d \],

then

\[ U = \text{Proj} T \quad \text{and} \quad f : \text{Proj} T \cong \text{Proj} S. \]

For this, use the fact that $\forall g \in S_+ \text{ homogeneous}$, and, $\forall n > 0$, $U_g = U_{g^n}$, $S[\varphi^{-1}] \cong S[\varphi^n]^{-1}$

and $S[\varphi^{-1}]_0 = S[\varphi^n]_0$.

Example 1: The d-relative embeddings $R$ a commutating

\[ P_n := P_R^n \quad m := \binom{n+d}{d} - 1 \]

counting exercise: $T := R[\gamma_0, \ldots, \gamma_n]$, the number of

degree d monomials in $\gamma_0, \ldots, \gamma_n$ is $\binom{n+d}{d}$.
Put $S := R[X_0, \ldots, X_n]$

Define a morphism of graded $R$-algebras $\phi: S \to T$ by first choosing an ordering of all the monomials of degree $d$ in $X_0, \ldots, X_n$ and sending $X_i$ to the $i$-th monomial. Using \textit{ex. II.2.14}, we obtain a morphism

$$f: \text{Proj } T \to \text{Proj } S$$

One can show

$$\mathbb{P}^n \to \mathbb{P}^n$$

closed embedding

\textit{Def:} this is the $d$-uple embedding.

A little more concretely, in terms of homogeneous
coordinates, I send \((b_0, \ldots, b_n) \in \mathbb{P}^n\) to the point of coordinates \((a_0, \ldots, a_n) \in \mathbb{P}^n\) where \(a_i\) is the \(i\)-th monomial of degree \(d\) in \(b_0, \ldots, b_n\).

**First case:** \(R = k\) alg. closed field

1. \(n = 1, \; d = 2\)
   
   \(m = \binom{1+2}{1} - 1 = 2\)

   \(\mathbb{P}^1 \rightarrow \mathbb{P}^2\) image is a conic

   \((b_0, b_1) \mapsto (a_0, a_1, a_2) = (b_0^2, b_0 b_1, b_1^2)\)

   relation: \(a_0 a_2 - a_1^2 = 0\)

   \(
   \Rightarrow \text{image } \subset \mathbb{Z} \left( x_0 x_2 - x_1^2 \right)
   
   \text{can show } = .\)
(2) Twisted cubic: \( u = 1 \), \( d = 3 \), \( m = \left( \frac{1 + 3}{3} \right) - 1 = 3 \)

\[ f: \mathbb{P}^1 \rightarrow \mathbb{P}^3 \]

\((v_0, v_1) \mapsto (a_0, a_1^2, a_2, a_3) = (v_0^3, v_0^2 v_1, v_0 v_1^2, v_1^3)\)

relations: \( a_0 a_3 - a_1 a_2, a_0 a_2 - a_1^2, a_1 a_3 - a_2^2 \)

\[ f(\mathbb{P}^1) \subset \mathbb{Z} (X_0 X_3 - X_1 X_2, X_0 X_2 - X_1^2, X_1 X_3 - X_2^2) \]

can show \( = \).