

The fact that  $\{\mathcal{O}_{\mathbb{P}^n}(k), k \in \mathbb{Z}\} / \sim \xrightarrow{\cong} \mathbb{Z}$

follows from  $H^0(\mathcal{O}_{\mathbb{P}^n}(k)) \cong S_k \subset S$

and  $\mathcal{O}_{\mathbb{P}^n}(k)^{-1} \cong \mathcal{O}_{\mathbb{P}^n}(-k)$ :

if  $k \neq k'$ , then  $\mathcal{O}_{\mathbb{P}^n}(k) \not\cong \mathcal{O}_{\mathbb{P}^n}(k')$

if  $k, k' > 0$ , then  $S_k \neq S_{k'} \Rightarrow \mathcal{O}_{\mathbb{P}^n}(k) \not\cong \mathcal{O}_{\mathbb{P}^n}(k')$

if  $k > 0, k' < 0$ , then  $H^0(\mathcal{O}_{\mathbb{P}^n}(k)) \neq 0$  and  $H^0(\mathcal{O}_{\mathbb{P}^n}(k')) = 0$

if  $k, k' < 0$ , then  $\mathcal{O}_{\mathbb{P}^n}(k)^{-1} \cong \mathcal{O}_{\mathbb{P}^n}(-k)$

$\not\cong \mathcal{O}_{\mathbb{P}^n}(-k') = \mathcal{O}_{\mathbb{P}^n}(k')^{-1}$

now use  $(\mathcal{L}^{-1})^{-1} \cong \mathcal{L}$  to conclude  $\mathcal{O}_{\mathbb{P}^n}(k) \not\cong \mathcal{O}_{\mathbb{P}^n}(k')$

More generally, for any locally free sheaf  $\mathcal{F}$  of finite rank:

$$(\mathcal{F}^*)^* \cong \mathcal{F}$$

## More operations on sheaves of modules:

Given a morphism of (noetherian) schemes  $f: X \rightarrow Y$   
and an  $\mathcal{O}_X$ -module  $\mathcal{F}$ ,  $f_* \mathcal{F}$  is  $(f_* \mathcal{O}_X)$ -module.

Using  $f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$  morphism of sheaves of rings.

this gives a structure of  $\mathcal{O}_Y$ -algebra on  $f_* \mathcal{O}_X$ .

So we can think of  $f_* \mathcal{F}$  as an  $\mathcal{O}_Y$ -module, using  $f^\#$ :  
we call this restriction of scalars.

The affine case: If  $X = \text{Spec } A$ ,  $Y = \text{Spec } B$

$M$  an  $A$ -module

$\mathcal{F} := \tilde{M}$  on  $X$ , then

$f_* \mathcal{F} = \tilde{M}$  on  $Y$  where  $M$  is  $M$  regarded as  
 $f_*$   $B$  a module over  $B$  via  $f^\# : B \rightarrow A$ .

A few words about proving this:

Given a basic open set  $\text{Spec } B[g^{-1}] \subset Y = \text{Spec } B$ ,  
we saw (last quarter)  $f^{-1}(\text{Spec } B[g^{-1}]) = \text{Spec } A[\bar{g}^{-1}]$   
where  $\bar{g} := f^\#(g) \in A$

$$\begin{aligned} \Rightarrow (f_* \mathcal{O}_f)(\text{Spec } B[g^{-1}]) &= \mathcal{O}_f(f^{-1}(\text{Spec } B[g^{-1}])) \\ &= \mathcal{O}_f(\text{Spec } A[\bar{g}^{-1}]) \\ &= M[\bar{g}^{-1}] =_B M[g^{-1}] \end{aligned}$$

$$\Rightarrow f_* \mathcal{O}_f \cong_B \widetilde{M}$$

Also  $\Rightarrow f_*$  (quasi-coherent) = quasi-coherent.

Pull-backs:

First, general inverse images:

For a continuous map  $f: X \rightarrow Y$  of topological spaces and a sheaf  $\mathcal{G}$  on  $Y$ , the inverse image sheaf  $f^{-1}\mathcal{G}$  on  $X$  is the sheaf associated to the presheaf

$$U \longmapsto \varinjlim_{V \supset f(U)} \mathcal{G}(V) \quad \text{if } f(U) \text{ is open} \\ = \mathcal{G}(f(U))$$

For any subset  $Z \xrightarrow{i} Y$ , the restriction of  $\mathcal{G}$  to  $Z$  is

$$\mathcal{G}|_Z := i^{-1}\mathcal{G}.$$

The stalk of  $\mathcal{G}|_Z$  at any  $P \in Z$  can be naturally identified with  $\mathcal{G}_P$ .

For a morphism of (noetherian) schemes  $f: X \rightarrow Y$  and a sheaf  $\mathcal{G}_Y$  of  $\mathcal{O}_Y$ -modules, the sheaf  $f^{-1}\mathcal{G}_Y$  is an  $(f^{-1}\mathcal{O}_Y)$ -module on  $X$ .

Note:  $f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$

also induces  $f^\# : f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$  (exercise)

The pull-back of  $\mathcal{G}_Y$  to  $X$  is

$$f^* \mathcal{G}_Y := f^{-1} \mathcal{G}_Y \otimes_{f^{-1} \mathcal{O}_Y} \mathcal{O}_X$$

Exercise (see homework): There is a natural isomorphism

$$\text{Hom}_{\mathcal{O}_X}(f^* \mathcal{G}_Y, \mathcal{F}) \cong \text{Hom}_{\mathcal{O}_Y}(\mathcal{G}_Y, f_* \mathcal{F})$$

for all  $\mathcal{O}_X$ -modules  $\mathcal{F}$  and all  $\mathcal{O}_Y$ -modules  $\mathcal{G}_Y$ .

In other words, the functors  $f^*$  and  $f_*$  are adjoint.

The affine case:  $X = \text{Spec } A$ ,  $Y = \text{Spec } B$

$N$  a  $B$ -module,  $\mathcal{N} := \tilde{N}$

then (exercise, remind the definitions):

$$f^* \tilde{N} \cong \left( N \otimes_B A \right)$$

↑ extension of scalars.

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More about quasi-coherent sheaves:  $X$  noetherian scheme.

$\mathcal{F}$  quasi-coherent on  $X$ .

By def.,  $\exists$  open cover of  $X$  by open affine schemes

$U = \text{Spec } A$  s.t.  $\exists$   $A$ -module  $M$  with  $\mathcal{F}|_U \cong \tilde{M}$ .

Theorem: If  $\mathcal{F}$  is quasi-coherent on  $X$  (noetherian),  
 then  $\forall$  open affine  $\text{Spec } B \subset X$ ,  $\exists$   $B$ -module  $N$   
 s.t.  $\mathcal{F}|_{\text{Spec } B} \cong \tilde{N}$ .

The main ingredient of the proof is the following

Lemma (5.3 in Hartshorne):  $X = \text{Spec } A$ ,  $\mathcal{F}$

quasi-coherent on  $X$ ,  $f \in A$ , then  
 (a) if  $s \in \Gamma(X, \mathcal{F})$  s.t.  $s|_{D(f)} = 0$

then  $\exists n > 0$  s.t.  $f^n s = 0 \in \Gamma(X, \mathcal{F})$

(b) if  $t \in \mathcal{F}(D(f)) = \Gamma(D(f), \mathcal{F})$ , then

$\exists n > 0$  s.t.  $f^n t$  is the restriction of a global section of  $\mathcal{F}$  over  $X$ .

Proof of the lemma:  $\exists$  open covering of  $X = \text{Spec} A$

by open affine sets  $\text{Spec} B$  s.t.  $\exists$   $B$ -mod.  $N$  with  $\mathcal{O}_{\text{Spec} B} \cong \tilde{N}$ .

We can write  $\text{Spec} B = \bigcup_{\text{some } g \in A} \text{Spec} A[g^{-1}]$   
 $X = \text{Spec} A$

$$\begin{array}{ccc} g \in A & \longrightarrow & B \\ g & \longmapsto & \bar{g} \in B \end{array} \longrightarrow A[g^{-1}] \underset{\text{universal prop. of localizations}}{=} B[\bar{g}^{-1}]$$

$$\mathcal{O}_{\text{Spec} B} \cong \tilde{N} \implies \mathcal{O}_{\text{Spec} B[\bar{g}^{-1}]} \cong \widetilde{N[\bar{g}^{-1}]}$$

$$\begin{array}{c} \text{SII} \\ \mathcal{O}_{\text{Spec} A[g^{-1}]} \end{array}$$

$$N[\bar{g}^{-1}] = N \otimes_B B[\bar{g}^{-1}] \cong N \otimes_B A[g^{-1}] =: M \text{ module over } A[g^{-1}]$$



$$\exists \mathcal{O}_X \big|_{\text{Spec } A[g^{-1}]} \cong \tilde{M}$$

$$\Rightarrow X = \bigcup_{\text{some } g \in A} \text{Spec } A[g^{-1}] \text{ s.t. } \forall g \mathcal{O}_X \big|_{\text{Spec } A[g^{-1}]} \cong \tilde{M}$$

$X$  affine  $\Rightarrow X$  quasi-compact

$$\Rightarrow X = \bigcup_{i=1}^n \text{Spec } A[g_i^{-1}]$$

$$\text{s.t. } \forall i \mathcal{O}_X \big|_{\text{Spec } A[g_i^{-1}]} \cong \tilde{M}_i \quad M_i \text{ module } A[g_i^{-1}]$$

$\mathcal{O}_X$  part (a) :  $s \in \mathcal{O}_X(X)$  s.t.  $s \big|_{\text{Spec } A[f^{-1}]} = 0$

$$\forall i \quad s \big|_{D(f) \cap D(g_i)} = 0$$

$$D(f) \cap D(g_i) = D(g_i f) = \text{Spec } A[g_i f^{-1}]$$

$$\mathcal{O}_f \Big|_{\text{Spec } A[g_i f^{-1}]} \cong \widetilde{M}_i \Rightarrow \mathcal{O}_f(D(g_i f)) = M_i[f^{-1}]$$

$$s_i := s \Big|_{D(g_i) = \text{Spec } A[g_i^{-1}]} \in M_i$$

$$s_i \Big|_{D(g_i f)} = s \Big|_{D(g_i) \cap D(f)} = 0 \qquad s \Big|_{D(g_i f)} = s \Big|_{D(g_i f)} \in M_i[f^{-1}]$$

$$\begin{aligned} & s_i \Big|_{\text{Spec } A[g_i f^{-1}][f^{-1}]} = 0 \\ \Rightarrow & \exists n_i > 0 \quad \text{s.t.} \quad f^{n_i} s_i = 0 \end{aligned}$$

part  $w := \max_{i=1, \dots, n} \{m_i\} \Rightarrow \int^w s \Big|_{D(g_i)} = 0 \quad \forall i$

$\bigcup_{i=1}^n D(g_i) = X \Rightarrow \int^w s = 0$  by the sheaf axioms.

For part (b):  $X = \bigcup_{i=1}^n D(g_i), \quad D(f) = \bigcup_{i=1}^n D(g_i, f)$

$t \in \Gamma(D(f), \mathcal{O}_f)$

$t \Big|_{D(g_i, f)} \in \Gamma(D(g_i, f), \mathcal{O}_f) = M_i[f^{-1}]$

$\Rightarrow \exists \begin{cases} m_i \in M_i \\ u_i > 0 \end{cases} \quad t \Big|_{D(g_i, f)} = \frac{m_i}{f^{u_i}}$

$\Rightarrow \int^{m_i} t \Big|_{D(g_i, f)} = m_i \Big|_{D(g_i, f)}$  after possibly increasing  $m_i$  and

replacing  $u_i$  by  $f^{r_i} u_i$  for some  $r_i \geq 0$

take  $n := \max_{i=1, \dots, n} \{r_i\}$  then  $f^n t \Big|_{D(g_i f)}$  is the

restriction of some  $u_i \in \Gamma(D(g_i), \text{etc})$   
(might again have to replace  $u_i$  with some  $f^{h_i} u_i$ )

Final step: we need to glue these  $u_i$  to a global section of  $\mathcal{O}_E$ .

$$u_i \Big|_{D(g_i g_j)} \stackrel{?}{=} u_j \Big|_{D(g_i g_j)} \quad \forall i, j$$

consider

$$\left( \begin{array}{c} \mu_i \\ \hline D(g_i, g_j) \end{array} - \begin{array}{c} \mu_j \\ \hline D(g_i, g_j) \end{array} \right) \Big|_{D(g_i, g_j, f)} = 0$$

because

$$\mu_i \Big|_{D(g_i, g_j, f)} = \begin{pmatrix} 1 & \dots & 1 \\ \hline \end{pmatrix} \Big|_{D(g_i, g_j, f)} = \mu_j \Big|_{D(g_i, g_j, f)}$$

we apply part (a):  $\exists \mu_{ij} > 0$  s.t.

$$\mu_{ij} \left( \begin{array}{c} \mu_i \\ \hline D(g_i, g_j) \end{array} - \begin{array}{c} \mu_j \\ \hline D(g_i, g_j) \end{array} \right) = 0 \quad \text{in } \Gamma(D(g_i, g_j), \mathcal{P})$$

$$\mu := \max_{i,j} \{ \mu_{ij} \}$$

$$\Rightarrow \forall i, j \quad \begin{pmatrix} \mu & \dots & \mu \\ \hline \end{pmatrix} \Big|_{D(g_i, g_j)} = \begin{pmatrix} \mu & \dots & \mu \\ \hline \end{pmatrix} \Big|_{D(g_i, g_j)}$$

$\Rightarrow \exists s \in \Gamma(X, \mathcal{O}_X) \text{ s.t.}$

$$s|_{D(g_i)} = f^u u_i \quad \forall i$$

$$\forall i \quad (f^{u+n} t)|_{D(g_i)} = (f^u u_i)|_{D(g_i)}$$

$$\Rightarrow f^{u+n} t = s|_{D(f)} = s|_{D(f)}$$

□