Sheaves of modules:

Throughout, X will denote a noetherian scheme and $\mathcal{O}_X$ will denote its structure sheaf. For a more general setting, see EGA (Elements de Géometrie algébrique)

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(1) A sheaf of $\mathcal{O}_X$-modules is a sheaf $\mathcal{F}$ on X s.t. $\forall$ open $U \subset X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module s.t. $\forall$ open sets $V \subset U \subset X$ and $a \in \mathcal{O}_X(U)$, $s \in \mathcal{F}(U)$, we have $(as)\big|_V = a\big|_V \, s\big|_V$

(2) A morphism of $\mathcal{O}_X$-modules $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves s.t. $\forall$ open $U \subset X$, $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules.

(3) The kernel, cokernel and image of a morphism of $\mathcal{O}_X$-module is an $\mathcal{O}_X$-module. The quotient of two $\mathcal{O}_X$-modules is an $\mathcal{O}_X$-module: cokernel of an inclusion.

(4) The direct sums, direct products, direct limits and inverse limits of $\mathcal{O}_X$-modules are again $\mathcal{O}_X$-modules. (see exercises in Chapter II. Section 1)

(5) Tensor products: Given $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$, the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheaf associated to the presheaf

$$U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$$

(this presheaf is seldom a sheaf)

(6) <u>The sheaf $\mathcal{H}om$</u>: Given $\mathcal{F}, \mathcal{G}$, the set

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) := \left\{ \begin{array}{c} \text{morphisms of } \mathcal{O}_X\text{-modules} \\ \text{from } \mathcal{F} \text{ to } \mathcal{G} \end{array} \right\}$$

is a modules over $\mathcal{O}_X(X)$.

the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the sheaf

$$U \longmapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$$

which is also an $\mathcal{O}_X$-module.

(7) An $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if there exists an open cover of X by affine open subschemes $U = \mathrm{Spec}\, A$ s.t. $\exists$ an A-module M with $\mathcal{F}|_U \cong \tilde{M}$

recall: $\tilde{M}$ is the sheaf on $\mathrm{Spec}\, A$ s.t. $\forall$ basic open $\mathrm{Spec}\, A[f^{-1}]$, $\tilde{M}(\mathrm{Spec}\, A[f^{-1}]) = M[f^{-1}]$.

One can prove that then $\forall$ open affine $V = \operatorname{Spec} B \subset X$, $\exists$ B-module N s.t. $\mathcal{F}|_V \cong \tilde{N}$.

A quasi-coherent sheaf is called coherent if, in addition, the modules M are finite.

(8) An $\mathcal{O}_X$-module is called free if it is isomorphic, as an $\mathcal{O}_X$-module, to a direct sum of sheaves isomorphic to $\mathcal{O}_X$. It is locally free if X has an open covering by open sets U s.t. $\mathcal{F}|_U$ is free.

<u>Note</u>: Locally free sheaves are quasi-coherent:

$$\mathcal{O}_X^{\oplus I}|_{\operatorname{Spec} A} = \widetilde{\left(A^{\oplus I}\right)}$$

An isomorphism $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus I}$ is called a trivialization of $\mathcal{F}$ on U.

The rank of a locally free sheaf is the number of copies of $\mathcal{O}_U$ in any trivialization, it can be finite or infinite. The rank of a locally free sheaf is constant on each connected component of X.

Locally free sheaves are coherent when they have finite rank.

<u>Glueing data:</u> Suppose $\mathcal{F}$ is locally free of rank n.

$\exists \quad X = \bigcup_{i=1}^{m} U_i \qquad$ (recall X is noetherian)

s.t. $\forall i \qquad \exists i \qquad \mathcal{F}|_{U_i} \xrightarrow[\varphi_i]{\cong} (\mathcal{O}_{U_i})^{\oplus n}$

$\mathcal{O}_n$ $\quad U_i \cap U_j$:

$$\begin{tikzcd} \mathcal{O}_{U_i\cap U_j}^{\oplus n} \arrow[d, "\varphi_j \circ \varphi_i^{-1}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{E}|_{U_i\cap U_j} \arrow[l, "\varphi_i"'] \arrow[d, equal, "\mathrm{Id}"] \\ \mathcal{O}_{U_i\cap U_j}^{\oplus n} & \mathcal{E}|_{U_i\cap U_j} \arrow[l, "\varphi_j"'] \end{tikzcd}$$

<u>def</u>: the glueing data is

$$\left\{ \varphi_j \circ \varphi_i^{-1} : \mathcal{O}_{U_i\cap U_j}^{\oplus n} \circlearrowleft \right\}$$

$$\varphi_j \circ \varphi_i^{-1} \longleftrightarrow \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \in GL_{\mathcal{O}_{U_i\cap U_j}}\left(\mathcal{O}_{U_i\cap U_j}^{\oplus n}\right)$$

$$a_{ij} \in \mathcal{O}_X(U_i \cap U_j)$$

the matrix (a_{kl}) is called a transition matrix.

(9) An invertible sheaf $\mathcal{L}$ is a locally free sheaf of rank 1.

Claim: If we put $\mathcal{L}^{-1} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$, then there is a natural isomorphism $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \cong \mathcal{O}_X$.

More generally, for any $\mathcal{O}_X$-module, $\exists$ a homomorphism

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \longrightarrow \mathcal{O}_X$$

on any U:
$$\mathcal{F}(U) \otimes Hom_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{O}_U) \longrightarrow \mathcal{O}_X(U)$$
$$s \otimes l \longmapsto l(s)$$

defines a morphism of presheaves, it factors through the sheaf $\mathcal{F} \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$

Notation: People often write $\mathcal{F}^* := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

When $\mathcal{F}$ is locally free of rank 1, one shows that this homomorphism is an isomorphism of $\mathcal{O}_X$-modules.

In fact, one can show that $\otimes_{\mathcal{O}_X}$ defines a group structure on the set $\{\text{invertible sheaves}\}/\sim$: this is the Picard group of X, denoted $\mathrm{Pic}(X)$. (abelian)

<u>Important example:</u> Fix an algebraically closed field k.

$$X := \mathbb{P}^n_k = \mathrm{Proj}\, k[X_0, \dots, X_n]$$

For any $m \in \mathbb{Z}$, we define an invertible sheaf $\mathcal{O}_{\mathbb{P}^n}(m)$.

$$\mathbb{P}^n_k := \bigcup_{i=0}^{n} U_i \qquad U_i = \mathrm{Spec}\, k\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right]$$
$$= \{\mathfrak{p} \mid \mathfrak{p} \not\ni X_i\}$$

On U_i, define $\mathcal{O}_{U_i}(m) := \mathcal{O}_{U_i} \underbrace{X_i^m}_{\text{formal generator}}$

we glue $\mathcal{O}_{U_i}(m)$ with $\mathcal{O}_{U_j}(m)$ on $U_i \cap U_j$:

$$\begin{tikzcd}
\mathcal{O}_{U_i\cap U_j} & \mathcal{O}_{U_i\cap U_j}(m) = \mathcal{O}_{U_i\cap U_j}\cdot X_i^m \arrow[l, "\cong"', "\varphi_i"] \arrow[d, equal] \\
& \mathcal{O}_{U_i\cap U_j}(m) = \mathcal{O}_{U_i\cap U_j}\cdot X_j^m \arrow[d, "\cong"', "\varphi_j"] \\
& \mathcal{O}_{U_i\cap U_j}
\arrow[from=1-1, to=3-2]
\end{tikzcd}$$

$$\cdot \left(\frac{X_j}{X_i}\right)^m$$

$$\varphi_j \circ \varphi_i^{-1} = \left(\frac{X_j}{X_i}\right)^m$$

glue $\mathcal{O}_{U_i}$ to $\mathcal{O}_{U_j}$ using $\left(\frac{X_j}{X_i}\right)^m$

$\Rightarrow$ well-defined sheaf on $\mathbb{P}^n$, called $\mathcal{O}_{\mathbb{P}^n}(m)$.

<u>Exercise</u>: $\mathcal{O}_{\mathbb{P}^n}(-m) \cong \mathcal{O}_{\mathbb{P}^n}(m)^{-1} := \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{\mathbb{P}^n}(m), \mathcal{O}_{\mathbb{P}^n})$

In general, if $\mathcal{L}$ has trivializations $\mathcal{L}|_{U_i} \xrightarrow{\cong} \mathcal{O}_{U_i}$ with transition functions $f_{ij} := \varphi_j \circ \varphi_i^{-1}$ on $U_i \cap U_j$, then $\mathcal{L}^{-1}$ has trivializations $\mathcal{L}^{-1}|_{U_i} \xrightarrow[\varphi_i]{\cong} \mathcal{O}_{U_i}$ with transition functions $g_{ij} := (f_{ij})^{-1} \in \mathcal{O}_{U_i \cap U_j}$.

<u>Claim</u>: $\{\mathcal{O}_{\mathbb{P}^n}(m),\ m \in \mathbb{Z}\}/\sim \ \xleftarrow{\cong} \mathbb{Z}$

This will follow from:

<u>Claim</u>: $\Gamma(\mathcal{O}_{\mathbb{P}^n}(m)) \cong \{$homogeneous polynomials of degree $m\} \cup \{0\}$

<u>Proof</u>: Choose $s \in \Gamma(\mathcal{O}_{\mathbb{P}^n}(m)) = \mathcal{O}_{\mathbb{P}^n}(m)(\mathbb{P}^n)$

$\forall i \qquad s|_{U_i} \in \Gamma(\mathcal{O}_{\mathbb{P}^n}(m)|_{U_i}) = \Gamma(\mathcal{O}_{U_i} X_i^m)$

$$s|_{U_i} \in k\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right] X_i^m$$

$$\exists\ f_i\left(\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right) \quad \text{s.t.} \quad s|_{U_i} = f\left(\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right) X_i^m$$

$s|_{U_i}$ has to glue to $s|_{U_j}$ on $U_i \cap U_j$.

$$\Rightarrow \quad f_i\left(\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right) X_i^m = f_j\left(\frac{X_0}{X_j}, \dots, \frac{X_n}{X_j}\right) X_j^m$$

$\Rightarrow$ $\underbrace{f_i X_i^m}_{=: P_i}$ and $f_j X_j^m$ $=: P_j$ have no denominators.

polynomial and $P_i = P_j \quad \forall i, j$

Put $P := P_i = P_j$

define a map $\Gamma(\mathcal{O}_{\mathbb{P}^n}(m)) \longrightarrow S_m \subset S = k[X_0, \dots, X_n]$

$$s \longmapsto P.$$

To show this map is an isomorphism, we show it has an inverse:

Start with $P \in S_m$, $\forall i$, we can write

$$P = f_i\left(\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right) X_i^m \longrightarrow \text{factor } X_i^m$$

$$\downarrow$$

a section s_i of $\mathcal{O}_{U_i} \cdot X_i^m$

$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ by def. $\Rightarrow$ we get an inverse map $S_m \longrightarrow \Gamma(\mathcal{O}_{\mathbb{P}^n}(m))$.

$\square$