Preliminaries: \( k \) a field.

Recall that \( H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = S_d \) the vector space of homogeneous polynomials of degree \( d \) in \( n+1 \) variables over \( k \). Recall that each \( f \in S_d \) defines a hypersurface of degree \( d \) which is, by definition, the scheme of zeros \( Z(f) \) of a global section of the sheaf \( \mathcal{O}_{\mathbb{P}^n}(d) \).

We showed that the homogeneous ideal \( I_{Z(f)} \) is generated by \( f \), i.e., \( I_{Z(f)} = S f \subseteq S \), and in each affine chart \( U_i := D_+(x_i) \), the ideal \( I_{Z(f) \cap U_i} \) is generated by \( f \), i.e., \( I_{Z(f) \cap U_i} = k \left( \frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i} \right) \frac{f}{x_i} \subseteq k \left[ \frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i} \right] \).
The ideal dual of \( Z(f) \) is the image of

\[ \mathcal{O}_{\mathbb{P}^n} (-d) \xrightarrow{f} \mathcal{O}_{\mathbb{P}^n}. \]

The scheme \( Z(f) \) has an associated Weil (or Cartier) divisor which is

\[ \sum_{i=1}^{m} n_i [Z(f_i)] \]

where

\[ f = \prod_{i=1}^{m} f_i \]

is the decomposition of \( f \) into a product of irreducible factors.

The description above in terms of the affine covering shows that, given two homogeneous polynomials \( f, g \) of degree \( d \), we have \( Z(f) = Z(g) \) iff \( \exists r \in \mathbb{K}^* \) st.

\[ f = r g. \]
This is true more generally for any nonsingular projective irreducible variety $X/k$ (Prop. II.7.7) with $L \to (X$ embeds in some $\mathbb{P}^n_k$ as a closed subscheme $)$ invertible sheaf $L$ on $X$:

$$\forall s, t \in H^0(L)$$

$$Z(s) = Z(t) \implies \exists \alpha \in k^\times \text{ s.t. } t = \alpha s.$$ 

$$|L| := \text{Proj } H^0(L) := \{ \text{the set of lines in } H^0(L) \}$$

$$= H^0(L) / k^\times$$

is called the complete linear system of $L$.

Recall that any effective Weil divisor $D$ is the divisor of (the scheme of) zeros of $s$, for some $s \in H^0(C \times (D))$. 

$s$ is unique up to multiplication by a non-zero constant $c \in \mathbb{k}$. (The ideal sheaf $I_Z(s) \cong \mathcal{O}_X(-D)$).

Bézout's Theorem: $k$ an alg. closed field. $X$ a non-singular closed irreducible subvariety of $\mathbb{P}^n_k$. ($X \subset \mathbb{P}^n_k$)

Then the set of hyperplanes $H \subset \mathbb{P}^n_k$ such that $H \not\subset X$ and $X \cap H$ is non-singular is a non-empty Zariski open subset of $|\mathcal{O}_{\mathbb{P}^n_k}(1)| \cong \mathbb{P}^n_k$.

Proof: Consider the product $X \times_k |\mathcal{O}_{\mathbb{P}^n_k}(1)|$.

Recall (Ex. II.3.23) that the set of closed points of $X \times_k |\mathcal{O}_{\mathbb{P}^n_k}(1)|$ is the product of the set of closed points of $X$ and $|\mathcal{O}_{\mathbb{P}^n_k}(1)|$. 

Consider the subset of closed points

\[ B := \{ (x, H) \mid x \in H, \text{ either } x \text{ is a singular point of } X \cap H \text{ or } H \supset X \} \subset X \times |G_{T^n}(1)| \]

Claim: \( B \) has a natural structure of a closed subscheme of \( X \times |G_{T^n}(1)| \).

Assume the claim for a moment.

We show that \( B \neq X \times |G_{T^n}(1)| \).

Let \( f \) be an equation for \( H \) (\( f \in H^0(G_{T^n}(1)) \)).

We want to understand what it means for \( x \in X \) to be a singular point of \( X \cap H \), \( x \in X_{\mathbb{C}} \subset T_{P_k}^n \).

\[ \exists i : x \in U_i = D_+(X_i) \]
In $U_i$, the ideal of $H \cap U_i$ is generated by $\frac{f}{x_i}$.

In $G_{x_i, x}$, the ideal of $H \cap X$ is generated by the germ of $\frac{f}{x_i}$. The local ring $O_{X, x}$ is equal to

$$O_{X, x} / (\frac{f}{x_i})_x$$

$x \in H \Rightarrow (\frac{f}{x_i})_x \neq 0$ in $O_{X, x}$.

(Otherwise $H \cap$ neighborhood of $x$ $\Rightarrow H \supset X$ because $X$ is irreducible.)

(Recall from first quarter: $\dim O_{X, x} / (\frac{f}{x_i}) = \dim O_{X, x} - 1$)

Denote $d = \dim X = \dim G_{x, x}$

$x \in H$ is non-singular at $x$ $\Rightarrow$ the maximal ideal of $x$ in $G_{x, x}$ is generated by $d-1$ elements.
This means that \((\frac{f}{x_i})_x\) can be completed to a set of generators of \(m_x \subset O_{X, x}\). By Nakayama's lemma, this means \((\frac{f}{x_i})_x \not\equiv \text{mod } m_x^2\). (take the image in \(m_x^2/m_x^2\) and complete to a basis of \(m_x/m_x^2\) over \(k = \text{field} \).)

So we have that

\[ H^0(X, x) \] is a singular point of \(X \cap H\)

iff \(f_x\) is in the kernel of the \(k\)-linear map

\[ \varepsilon_x : H^0(C_{fu}(1)) \to \frac{O_{X, x}}{m_x^2} \to \frac{m_x}{m_x^2} \]

\[ f \to (\frac{f}{x_i})_x \mod m_x^2 \]
The map changes by multiplication by a nonzero \( \lambda \in k \) if we change the index \( i \), so the kernel is independent of the choice of \( i \).

\[
\text{Note: } \exists x \text{ is surjective, because } M_x \text{ is generated by}
\]

image of elements of \( H^0(\mathbb{P}^n, \mathcal{I}_0(1)) = kX_0 \oplus \cdots \oplus kX_n \)

\( x \) is a closed point of \( \mathbb{P}^n \) and of \( U_i \subseteq \mathbb{A}^n \)

\( x \mapsto \) maximal ideal of \( k \left[ \frac{X_0}{X_i}, \cdots, \frac{X_n}{X_i} \right] \)

\( \mapsto \) homogeneous maximal ideal of \( k[X_0, \cdots, X_n] \)

\[
= (b_i X_j - b_j X_i) \quad 0 \leq i < j \leq n
\]

And \( \exists g \in H^0(\mathbb{P}^n, \mathcal{I}_0(1)) \) s.t. \( \varepsilon_x(g) \notin M_x / M_x^2 \).