Example of the cuspidal cubic:

\[ C = \mathbb{k}[[x,y]] / (y^2 - x^3) \]

\[ (0,0) \in \text{Spec } \mathbb{C} \text{ singular point.} \]

\[ \Omega^1_C / \mathbb{k} = \frac{C dx \oplus C dy}{\langle d(y^2 - x^3) \rangle} \]

\[ = C dx \oplus C dy / \langle 2y \, dy - 3x^2 \, dx \rangle \]

Open set \( D(x) \): \( C = \left( \mathbb{k}[[x,y]] / (y^2 - x^3) \right)[x^{-1}] \quad \mathbb{k}[x^{-1}] = \mathbb{k} \)

\[ \Omega^1_C[x^{-1}] / \mathbb{k}[x^{-1}] \]

\[ = -\Omega^1_C / \mathbb{k} \]

\[ = C[x^{-1}] dx \oplus C[x^{-1}] dy \]

\[ \frac{(2y \, dy - 3x^2 \, dx)}{\langle 2y \, dy - 3x^2 \, dx \rangle} \]

\[ \approx C[x^{-1}] dy \quad \text{free.} \]
Similarly, $\Omega^1_{\mathbb{C}(g) \mathbb{P}^1}$ is free of rank $1/\mathbb{C}$.

This means (we will see this in more detail) that $C$ is smooth on the two open sets $D(x)$ and $D(y)$.

$(0,0) \notin D(x) \cup D(y)$. (char. $\neq 2,3$)

Sheafification: Suppose given a scheme $X$ over a scheme $S$.

Cover $S$ with open affine schemes $\text{Spec} A$ and $X$ with open affine schemes $\text{Spec} B$ s.t. $\text{Spec} B$ maps to $\text{Spec} A$.

For each $\text{Spec} B \to \text{Spec} A$, we define

$$\Omega^1_{\text{Spec } B/\text{Spec } A} := \Omega^1_{B/A}$$

quasi-coherent on $\text{Spec } B$.

These glue to a global sheaf $\Omega^1_{X/S}$ because taking relative differentials commutes with localization.
Last quarter: any intersection \( \text{Spec} A \cap \text{Spec} A' \) (resp. \( \text{Spec} B \cap \text{Spec} B' \)) can be covered with affine open sets that are basic for \( \text{Spec} A \) and \( \text{Spec} A' \) (resp. \( \text{Spec} B \) and \( \text{Spec} B' \)), i.e., of the form

\[
\text{Spec} A[f^{-1}] = \text{Spec} A'[f'^{-1}]
\]

(\( \Rightarrow \) \( \text{Spec} B[f^{-1}] = \text{Spec} B'[f'^{-1}] \)).

Then

\[
\Omega^1_{B/A}[f^{-1}] = \Omega^1_{B[f^{-1}]/A[f^{-1}]} = \Omega^1_{B'[f'^{-1}]/A'[f'^{-1}]} = \Omega^1_{B'/A}[f'^{-1}].
\]

Alternatively, we can define \( \Omega^1_X/5 \) using the diagonal embedding.
\[ S = \bigcup_{j \in J} \text{Spec} A_j \qquad \tau_1 : X \to S \]

\[ \tau_1^{-1}\left( \text{Spec} A_j \right) = \bigcup_{k \in K_j} \text{Spec} B_{kj} \]

\[ X \times X = \bigcup_{S} \text{Spec} B_{k1j} \otimes B_{k2j} \cup \bigcup_{A_j} \text{Spec} B_{k1j} \otimes A_j \]

\[ \Delta : X \to \bigcup_{A_j} \text{Spec} B_{k1j} \otimes A_j \hookrightarrow X \times X \]

\[ \Delta \quad \text{closed embedding} \quad \tau_0 \circ \Delta = \tau_2 \circ \Delta = \text{Id}_X \]

Because \[ \phi_1 \circ \Delta = \phi_2 \circ \Delta = \text{Id}_X \]

Let \( I \) be the ideal sheaf of the image of the diagonal in \( \bigcup_{A_j} \text{Spec} B_{k1j} \otimes B_{k2j} \). We can define \( \mathcal{L}' / L \cdot S = \Delta^*(\mathcal{L}/L_2) \).
In both case, we have a differential:

\[ d: \mathcal{O}_X \to \Omega^1_X \otimes_1. \]

obtained from the differentials \( B \to \Omega^1_B \otimes A \)

Properties:

1. Base change:

\[
\text{Spec} B' = \text{Spec} B \otimes_A A' = X \times_S S' \xrightarrow{\pi_1} X = \text{Spec} B \]
\[
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\]
\[
\text{Spec} A' = S' \longrightarrow S = \text{Spec} A
\]

\[ \Omega^1_{B'/A'} = \Omega^1_{B/A} \otimes B' \]

\[ \Rightarrow \quad \Omega^1_{B'/A'} \cong \Omega^1_{B/A} \otimes B' = \pi_1^* \Omega^1_{\text{Spec} B'/\text{Spec} A} \]
In the general case, these will glue to

\[ L_i^{\cdot} \times_S S' \cong \rho^* \pi_i^{\cdot} \times_S S \]

(2) **Pull-back**: \( A \to B \to C \)

\[ \text{Spec } C \to \text{Spec } B \to \text{Spec } A \]

\[ L_i^{\cdot} \otimes_{B/A} C \to L_i^{\cdot} \otimes_{C/A} C \to L_i^{\cdot} \otimes_{C/B} C \to 0 \]

\[ \Rightarrow \]

\[ L_i^{\cdot} \otimes_{B/A} C \to L_i^{\cdot} \otimes_{C/A} C \to L_i^{\cdot} \otimes_{C/B} C \to 0 \]

\[ p^* L_i^{\cdot} \to L_i^{\cdot} \]

In the general case, these glue to \(( Y \to X \to S )\)

\[ p^* L_i^{\cdot} \times_S S' \to L_i^{\cdot} Y/S \to L_i^{\cdot} Y'/X \to 0 \]
(3) Restriction to closed subschemes:

\[ A \to B \to C = B/I \]

\[ Y = \text{Spec } C \to \text{Spec } B \to \text{Spec } A = S \]

Sheafify:

\[ \frac{I/I^2}{I} \to \Omega^1_{B/A} \otimes C \to \Omega^1_{C/A} \to 0 \]

In general,

\[ \frac{I^2/I}{I^2} \to \frac{i^* \Omega^1_{\text{Spec } C/\text{Spec } A}}{i^* \Omega^1_{\text{Spec } B/\text{Spec } A}} \to \Omega^1_{\text{Spec } C/\text{Spec } A} \to 0 \]

\[ \frac{J_Y/\mathcal{J}_Y}{\mathcal{J}_Y} \to \frac{i^* \Omega^1_{X/S}}{X/S} \to \Omega^1_{Y/S} \to 0 \]

For \( Y \subset X \to S \) closed
Varieties: Definition: A variety over a field $k$ is a reduced separated scheme of finite type over $k$.

Note: Hartshorne also assumes varieties to be irreducible.

Definition: A variety $X/k$ is called non-singular if all its local rings are regular local rings.

Note: Any localization of a regular local ring is again regular (Theorem II.8.144). In other words, being regular is stable under localization.

The set of points of a scheme whose local rings are not regular is its singular locus.

The complement is the non-singular (a regular) locus.
So the singular locus is closed under specialization and the nonsingular locus is closed under
generation.
We shall see that if $X$ is a variety over a field, then
its singular locus is a proper closed subset.

Theorem (II.8.15) $X$ separated of finite type over an
algebraically closed field $k$. Then $\Omega^1_{X/k}$ is a
locally free sheaf of finite rank equal to the dimension
of $X$ if and only if $X$ is a nonsingular variety $/k$.
The same statement holds if $k$ is not necessarily
algebraically closed, provided that $X$ is a variety.