Differentials:

We want an algebraic way to define tangent vectors.

In differential geometry, tangent vectors act on functions.

And, at any point \( x \in M \) manifold, given a function \( f \) in a neighborhood of \( x \) and a tangent vector \( \nu \) at \( x \),

\[ \nu(f) = \nu(f - f(x)). \]

So we can concentrate on functions that vanish at \( x \).

Algebraically, given a scheme \( X \) and a point \( x \in X \),
we can think of \( M_x \subset O_X \) as the set of functions defined in a neighborhood of \( x \) which vanish at \( x \).

\[ M_x / M_x^2 \]
represents the set of functions locally defined at \( x \), vanishing at \( x \), up to first order.
Taking derivatives is a linear operation, so \((M_x^*/M_x^1)^*\) is the right definition for the Zawiński tangent space (dual with values in \(k(x)\)).

Differentials allow us to globalize this, to define tangent sheaves for instance.

Derivations are the algebraic analogues of derivatives:

**Definition:** Let \(A\) be a commutative ring, \(B\) a commutative \(A\)-algebra, \(M\) a \(B\)-module.

An \(A\)-derivation of \(B\) into \(M\) is an \(A\)-linear map \(d : B \to M\) s.t. \(\forall b, b' \in B\),

\[
d(bb') = bb'd(b') + b'd(b).
\]
The set $\text{Der}_A(B,M)$ of all $A$-derivations of $B$ into $M$ has a natural structure of $B$-module defined by $(b \cdot d)(b') = b \cdot (d(b')) + b, b' \in B, d \in \text{Der}_A(B,M)$. We have a (covariant) functor

$$
\begin{array}{c}
\text{B-modules} \\ \longrightarrow \\ \text{B-modules}
\end{array}
\begin{array}{c}
M \\ \mapsto \\ \text{Der}_A(B,M)
\end{array}
$$

is representable by a $B$-module $\Omega^1_{B/A}$; by definition the module of relative differential forms of $B$ over $A$. The module $\Omega^1_{B/A}$ has its own canonical $A$-derivation:

$$d : B \longrightarrow \Omega^1_{B/A},$$

and it is characterized by the following universal property.
Universal property: \( + \) \( B \)-module \( M \) with \( A \)-derivation \( D: B \to M \), \( \exists ! \) hom. of \( B \)-modules \( h: \Omega^1_{B/A} \to M \) s.t. the diagram

\[
\begin{array}{ccc}
B & \xrightarrow{d} & \Omega^1_{B/A} \\
\downarrow \circ \downarrow & & \downarrow \circ \downarrow \\
D & \rightarrow & M
\end{array}
\]

commutes.

This guarantees uniqueness of \( \Omega^1_{B/A} \) up to isom.

We can show existence by constructing it (in two ways).

1) First way of constructing \( \Omega^1_{B/A} \):

\[ F := \text{ free } B \text{-module with basis } \{ d \cdot b \mid b \in B \}. \]
$\Omega^1_{B/A}$ can be defined as the quotient of $F$ by the submodule generated by
\[ \{ d(a b + a' b') - a d b - a' d b', \ d(b b') - b d b' - b' d b \mid a, a' \in A, b, b' \in B \} \]
$\alpha$, equivalently, the submodule generated by
\[ \{ d a, d(b b') - b d b' - b' d b \mid a \in A, b, b' \in B \} \]
The differential $d : B \to \Omega^1_{B/A}$ is induced by
$B \to F, b \mapsto db$.

2) The second way of constructing $\Omega^1_{B/A}$:
Let $m : B \otimes_A B \to B, b \otimes b' \mapsto b b'$ be the multiplication map, let $I := \langle 1 \otimes b - b \otimes 1 \mid b \in B \rangle \subset B \otimes B$
be the kernel of $m$. Then $\Omega^1_{B/A} = I / I^2$ and the
differential \( d : B \rightarrow \mathcal{O}_{B/A} = \mathfrak{I} \bigwedge B \)

via

\[ \downarrow \Rightarrow \mathfrak{I} \mathfrak{O}_B - \mathfrak{I} \mathfrak{O}_A = : \mathfrak{d} \mathfrak{I} \]

The idea here: If \( Y = \text{Spec} B, \ X = \text{Spec} A \).

\[ Y \times_Y Y = \text{Spec} B \otimes_A B \]

\[ \xrightarrow{\Delta} Y \]

\[ \text{diagonal} \]

\[ I = \text{ideal of the diagonal.} \]
$I$ is a $B^\otimes B^\otimes A$-module, so $I/I^2$ is a $(B^\otimes B/\underline{\Pi})$-module.

Can verify that at a point $y \in Y$,

\[(I/I^2)_y \cong m_y/m^2_y \text{ prime ideal} \]

**Example:**  $A = k$ a field, $B = k[x_1, \ldots, x_n]$

\[P \in k[x_1, \ldots, x_n] \quad \frac{dP}{dx} = \sum_{i=1}^n \frac{\partial P}{\partial x_i} \quad \in \mathbb{R}^n_{B/A} \quad \text{by the relations } da = 0 \text{ and } d(ll') = l'd + l'd'. \]

$\frac{\partial P}{\partial x_i}$ is the formal partial derivative with respect to $x_i$:

E.g.: $P = x_1^2 + x_2^3 - x_3 x_4$, $\frac{\partial P}{\partial x_1} = 2x_1$

$\Rightarrow \mathbb{R}^n_{B/A}$ is the free $B$-module with basis $\{dx_1, \ldots, dx_n\}$. 
Properties:

(1) **Base change:** Given another $A$-algebra $A'$, define $B':= B \otimes_A A'$. Then

$$\Omega^1_{B'/A'} = \Omega^1_{B/A} \otimes B' .$$

In particular, if $A'= S^{-1}A$ for a multiplicative set $S$, we have $B'= S^{-1}B$ and

$$\Omega^1_{S^{-1}B/S^{-1}A} = \Omega^1_{B/A} \otimes S^{-1}B = S^{-1} \Omega^1_{B/A} .$$

This will later allow us to simplify.

(2) **Pull-back:** Given a $B$-algebra $C$, there is a natural sequence of $C$-modules:

$$\Omega^1_{B/A} \otimes C \rightarrow \Omega^1_{C/A} \rightarrow \Omega^1_{C/B} \rightarrow 0$$

This is (2) above.
(3) Restriction to a closed subscheme: \( C = B/I \)

\[
A \to B \to B/I = C
\]

\[
\frac{I}{I^2} \xrightarrow{\delta} \Omega^1_{B/A} \otimes C \to \Omega^1_{C/A} \to 0
\]

where \( \delta(T) = db \otimes 1 \) for \( T \in I, \bar{T} \in I/I^2 \) the image of \( T \).

Example: \( A = k \) field, \( B = k[x_1, \ldots, x_n] \)

\[
I = \langle f_1, \ldots, f_n \rangle \subseteq B \text{ ideal}
\]

\[
\frac{\Omega^1_{B/A} \otimes C}{C-\text{module}} \quad \text{is the free } B-\text{module with basis } \{dx_1, \ldots, dx_n\}
\]

\[
\delta\left( \frac{f_i}{I} \right) = df_i \otimes 1 \quad \text{and} \quad df_i = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} \cdot dx_j
\]
\[ \Rightarrow \quad \frac{\partial}{\partial x_i} \in \mathfrak{c}^{1/k} \quad = \quad \frac{C \, dx_1 \otimes \cdots \otimes C \, dx_n}{\langle df_1, \ldots, df_n \rangle} \quad C = \frac{B}{\langle f_1, \ldots, f_n \rangle} \]

e.g.: \[ C = k(x, y) (y - x^2) \quad \text{the parabola.} \]

\[ \Rightarrow \quad \frac{\partial}{\partial x_i} \in \mathfrak{c}^{1/k} \quad = \quad \frac{C \, dx \otimes C \, dy}{d(y - x^2)} \quad d(y - x^2) = dy - 2x \, dx \]

\[ \Rightarrow \quad \mathfrak{c}^{1/k} \quad \subset \quad C \, dx \quad \text{is free of rank} \quad 1/C \quad \text{.} \]

\[ C = \frac{k(x, y)}{(y^2 - x^3)} \quad \text{the cuspidal cubic} \]