Any \( \eta \in X \) of codim. 1 s.t. \( \nu_\eta (f) < 0 \) is the generic point of one of the \( Y_i \).

\[ \{ \eta \in X \mid \eta \text{ of codim. 1, } \nu_\eta (f) < 0 \text{ is finite} \} \]

Replace \( f \) by \( f' \) to see that

\[ \{ \eta \in X \mid \eta \text{ of codim. 1, } \nu_\eta (f') > 0 \text{ is finite} \} \]

\( \Rightarrow \) \( \text{Div} (f) \) is well-defined. \( \square \)

**Definition:** The divisor class group \( Cl (X) \) is the quotient of the group \( \text{Div} (X) \) of Weil divisors on \( X \) by the subgroup generated by principal divisors.

Def: We say two Weil divisors \( D_1, D_2 \in \text{Div} (X) \) are linearly equivalent if their difference is principal, i.e.
There is a function \( f \in K \) s.t. \( D_1 - D_2 = \text{Div}(f) \).

Proof: If \( X = \text{Spec} \ A \) (in practice, \( A \) is a Noetherian integral domain), then \( A \) is a UFD if and only if \( X \) is normal (i.e., \( A \) is integrally closed) and \( \text{Cl}(X) = 0 \).

(Proof is Hartshorne)

Corollary: If \( X = \mathbb{A}^n_k \), then \( \text{Cl}(X) = 0 \)

Divisors in projective space: \( S := k[x_0, \ldots, x_n] = \bigoplus_{d \geq 0} S_d \)

\( X = \text{Proj } S = \mathbb{P}^n_k \). Let \( s \in S_d \), put \( Y := Z(s) \).

Lemma: \( Y \) is a closed subscheme of \( X \) of pure codimension 1 in \( X \) (i.e., every irreducible component of \( Y \) has codim 1).
The irreducible components of (the underlying topo. space of) \( Y \) are the zeros of the irreducible factors of \( s \).

Proof: The ideal sheaf \( \mathcal{I}_Y \) is the image of

\[
s : \mathcal{O}_X(-d) \rightarrow \mathcal{O}_X.
\]

Hence, in each open set \( U_i = \text{Spec} S[X_i^{-1}] \), the ideal sheaf \( \mathcal{I}_{Y \cap U_i} = \mathcal{I}_Y|_{U_i} = \mathcal{O}_{U_i} \frac{s}{X_i^d} \)

and

\[
\mathcal{I}_{Y \cap U_i} = H^0(\mathcal{I}_{Y \cap U_i}) = S[X_i^{-1}]_0 \cdot \frac{s}{X_i^d} \subset S[X_i^{-1}]_0.
\]

(recall \( \mathcal{I}_{Y \cap U_i} = \mathcal{I}_Y|_{U_i} \))
The homogeneous ideal of $Y$, $I_Y = \bigoplus_{e \geq 0} I_{Y,e}$

$I_{Y,e} = \{ t \in \mathbb{R} \mid Z(t) \subset \partial Y \}$

on $U_i$, this means $\forall t \in I_{Y,e}$,

$$\frac{t}{x_i^e} \in I_Y \cap U_i = \mathbb{S}(x_i^{-1}) \frac{s}{x_i^d}.$$?

$\Rightarrow \frac{t}{x_i^e}$ is a multiple of $\frac{s}{x_i^d}$

$\exists P_i \in \text{Hom. of degree } n_i \geq 0 \text{ s.t. } \frac{t}{x_i^e} = \frac{P_i}{x_i^{n_i}} \frac{s}{x_i^d}$

$\Rightarrow t = P_i x_i^{d+n_i-e} s \quad \forall i$.

$\Rightarrow t$ is a multiple of $s$. (exercise)
\[ \Rightarrow I_Y = \bigoplus_{e \geq 0} I_{Y,e} = SS = \text{(ideal generated by } s) \]

\[ s \in S_d \]

Write \( s = \sum_{i}^{m} s_i \) where \( s_i \) is irreducible of degree \( d_i \) and \( s_i, s_j \) are not proportional for \( i \neq j \).

\[ Z(s) = \bigcup Z(s_i) \text{ as a set. (} \cup Z(s_i) = SS_i \bigcup \]

\[ Z(s_i) \text{ is irreducible of codimension 1 in } X. \]

We determine \( \alpha(\text{P}^n_k = X) : \)

**General Lemma:** \( (X \text{ is not necessarily projective space}) \)

Suppose \( U \subset X \) is nonempty open and let \( Z_1, \ldots, Z_n \) be the codimension 1 irreducible components of \( X \setminus U \).
Then, intersecting divisors of \( X \) with \( U \) produces the exact sequence:

\[
0 \rightarrow \mathbb{Z}[Z_1] \oplus \cdots \oplus \mathbb{Z}[Z_n] \rightarrow \text{Div}(X) \rightarrow \text{Div}(U) \rightarrow 0
\]

which induces the exact sequence:

\[
\mathbb{Z}[Z_1] \oplus \cdots \oplus \mathbb{Z}[Z_n] \rightarrow \text{Cl}(X) \rightarrow \text{Cl}(U) \rightarrow 0
\]

**Proof:** The first exact sequence is immediate: a divisor of \( U \) is the intersection with \( U \) of its closure in \( X \).

For the second sequence, surjectivity is true for the same reason. For exactness in the middle; note that \( K = K(X) = K(U) \).

For \( f \in K \), \( (\text{Div}_X(f)) \cap U = \text{Div}_U(f) \).
\[ 0 \rightarrow \{ \text{Div}(f) \} \text{ supported on } X \setminus U \rightarrow \text{Prim}(X) \rightarrow \text{Prim}(U) \rightarrow 0 \]

\[ 0 \rightarrow \mathbb{Z}[\mathbb{Z}] \oplus \cdots \oplus \mathbb{Z}[\mathbb{Z}] \rightarrow \text{Div}(X) \rightarrow \text{Div}(U) \rightarrow 0 \]

\[ 0 \rightarrow \mathbb{Z}[\mathbb{Z}] \oplus \cdots \oplus \mathbb{Z}[\mathbb{Z}] \rightarrow \text{Cl}(X) \rightarrow \text{Cl}(U) \rightarrow 0 \]

\[ \text{If } D \in \text{Div}(X) \text{ satisfies } D \cap U = \text{Div}_U(f) \]

\[ \text{then } D - \text{Div}_X(f) = \text{combination of } [\mathbb{Z}], \ldots, [\mathbb{Z}] \]

\[ \Rightarrow \text{ the class of } D \text{ in } \text{Cl}(X) \in \text{subgroup generated by } [\mathbb{Z}], \ldots, [\mathbb{Z}] \]

\[ \Rightarrow \text{ exactness in the middle} \]
We use the lemma to compute $\mathbb{C}(\mathbb{P}^n)$.

Take $V = U_0$, $Z_0 := \mathbb{P}_k^n \setminus U_0$

\[ \mathbb{Z}[Z_0] \rightarrow \mathbb{C}(\mathbb{P}^n) \rightarrow \mathbb{C}(U_0) \rightarrow 0 \]

$U_0 \cong \mathbb{A}_k^m \Rightarrow \mathbb{C}(U_0) = \mathbb{0}$, so:

\[ \mathbb{Z}[Z_0] \rightarrow \mathbb{C}(\mathbb{P}^n) \]

Lemma: $\mathbb{Z}[Z_0] \rightarrow \mathbb{C}(\mathbb{P}^n)$ is injective, hence \[ \mathbb{Z}[Z_0] \cong \mathbb{C}(\mathbb{P}^n) \]

Proof: Injectivity means that there are no rational functions $f$ on $\mathbb{P}^n$ with $\text{Div}(f) = \text{multiple of } [Z_0]$. Choose $f \in K(\mathbb{P}^n) = K(U_0) = \text{Frac} k[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}]$

\[ = k(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) \]