Can we relate $\Gamma_\star(\mathcal{F}_0)$ to $\mathcal{F}_0$?

Note: There is always a natural morphism

$$\Gamma_\star(\mathcal{F}_0) \to \mathcal{F}_0.$$ 

This is defined on basic open sets as follows

$$f \in S_+ \text{ homogeneous}.$$ 

$$H^0(U(f), \Gamma_\star(\mathcal{F}_0)) \ (\coloneqq \Gamma(U(f), \Gamma_\star(\mathcal{F}_0))) = \Gamma_\star(\mathcal{F}_0)(f) = \Gamma_\star(\mathcal{F}_0)[f^{-1}] \to \mathcal{F}_0(U(f))$$

Given $s \in \Gamma_\star(\mathcal{F}_0)(f)$, there exists $t \in \Gamma_\star(\mathcal{F}_0)$

and $d > 0$ s.t. $s = \frac{t}{f^d}$.

$s$ has degree 0 $\Rightarrow$ $t \in \Gamma(\text{Proj } S, \mathcal{F}_0(d \cdot \text{ degree } f))$. 
\textit{but} \quad m = \text{degree}(f) \quad \text{so that} \quad t \in H^0(\text{Proj} S, \mathcal{O}(dm)).

The image of \( s \) in \( \Gamma(U(p), \mathcal{F}) \) is, by def., the image of \( t \otimes f^d \) via the tensor product map

\[ H^0(U(p), \mathcal{F}(dm)) \otimes H^0(U(q), \mathcal{O}(-dm)) \rightarrow H^0(U(q), \mathcal{F}(dm)\mathcal{O}(-dm)) = H^0(U(q), \mathcal{F}). \]

\textbf{Prop. II.5.15:} \textit{When} \( S \) \textit{is finitely generated by} \( S_1 \) \textit{as an} \( S_0 \)-\textit{algebra and} \( \mathcal{F} \) \textit{is quasi-coherent, this morphism is an isomorphism.}
Some important definitions:

**Def 1:** For any scheme $X$, $\tilde{P}_X := \tilde{P}^n_t \times X \times \text{Spec } \mathbb{Z}.$

$(X \to \text{Spec } \mathbb{Z}, \text{ obtained from } \mathbb{Z} \to H^0(X, \mathcal{O}_X))$

The twisting sheaf $\mathcal{O}_{\tilde{P}_X}^n(1)$ is, by definition,

$\mathcal{O}_{\tilde{P}_X}^n(1) := \mathcal{O}_{\tilde{P}^n_t} \otimes_{\mathcal{O}_Z} \mathcal{O}_{\tilde{P}_Z}^n(1)$

where $\mathcal{O}_P^t : \tilde{P}_X \to \tilde{P}^n_t$ is the first projection.

**Def 2:** A morphism of schemes $W \to X$ is an embedding if it is a composition

$W \to Z \hookrightarrow X$

where $W \to Z$ is an open embedding and $Z \hookrightarrow X$ is a closed embedding.
Def 3: For any morphism of schemes $X \rightarrow Y$, an invertible sheaf $L$ on $X$ is very ample relative to $Y$ if $\exists$ an embedding $i : X \hookrightarrow \mathbb{P}^n_Y$ s.t. $L = i^* \mathcal{O}_{\mathbb{P}^n_Y}(1)$.

Def 4: Given a sheaf $F$ and a collection $\{s_i, i \in I\}$ of global sections of $F$ on $X$, we say that $F$ is generated by $\{s_i, i \in I\}$ if, $\forall x \in X$, the stalk $F_x$ of $F$ at $x$ is generated by the germ of the $s_i$ at $x$ as an $\mathcal{O}_{X, x}$-module. We say that $F$ is generated by global sections, or globally generated, if it is generated by some collection of global sections.
Remark: We can define a morphism of sheaves:

\[ \varphi : \mathcal{O}_X \to \mathcal{O}_F \]

\[ \mathcal{G}(U) \ni \sum_{i \in I} f_i \longmapsto \sum_{i \in I} f_i \cdot s_i |_U \in \mathcal{F}(U) \]

finite sum

\[ \mathcal{F} \text{ is generated by } \{ s_i, i \in I \} \implies \varphi \text{ is surjective.} \]

Zeros of global sections of sheaves:

Let \( \mathcal{F} \) be a (quasi-)coherent sheaf on a noetherian scheme \( X \) and \( s \in H^0(X, \mathcal{F}) \). We define the scheme zero of \( s \), denoted \( Z(s) \), as follows:

\( \mathcal{Z} \) (closed subscheme of \( X \))
\[ s \in H^0(X, \mathcal{F}) \]
defines a morphism of \( O_X \)-modules
\[ s : \mathcal{F} \to O_X \]
\[ \mathcal{F}^*(U) \subseteq \mathcal{O}_U \]
\[ \mathcal{F}^* := \text{Hom}_{\mathcal{O}_X} (\mathcal{F}, \mathcal{O}_X) \]
\[ \mathcal{F}^* (U) = \text{Hom}_{\mathcal{O}_U} (\mathcal{F}_{\mid U}, \mathcal{O}_U) \]
The image of \( s \) is a coherent sheaf of ideals whose associated closed subscheme is, by def., \( Z(s) \).

Claim: The support \( \text{Supp} Z(s) \) of \( Z(s) \) (the underlying closed subset of \( X \)) is the set of points \( x \in X \)
st. the image of $s_x : F_x^{*} \to G_{X, x}$ is contained in the maximal ideal $\mathfrak{m}_x \subset G_{X, x}$.

$s : F^{*} \to G_X$

\[ Z(s) \]

\[ 0 \to Z(s) \to G_X \overset{i_x}{\to} G_{Z(s)} \to 0 \]

where $i : Z(s) \subset X$

$x \in Z(s) \iff \left( i_x \circ G_{Z(s)} \right) \neq 0$

(\[ \iff \mathcal{O}_{Z(s), x} \not\cong \mathcal{O}_{X, x} \]

(\[ \iff \mathcal{O}_{Z(s), x} \subset \mathfrak{m}_x \]

(\[ \iff s (F_x^{*}) \subset \mathfrak{m}_x \]

(\[ \iff s (F_x^{*}) \subset \mathfrak{m}_x \]
**Claim 2:** Suppose \( f \) is locally free. For \( x \in X \), \( \mathcal{F}_x \) is a free \( \mathcal{O}_{x, x} \) -module of finite rank: \( \mathcal{F}_x \cong \mathcal{O}_{x, x} \oplus \mathcal{O}_{x, x} \oplus \cdots \)

\[
\mathcal{F}_x^* = \text{Hom}_{\mathcal{O}_{x, x}}(\mathcal{F}_x, \mathcal{O}_{x, x}) \cong \text{Hom}_{\mathcal{O}_{x, x}}(\mathcal{O}_{x, x} \oplus \cdots \mathcal{O}_{x, x} \oplus \cdots, \mathcal{O}_{x, x}) \cong \mathcal{O}_{x, x} \oplus \cdots \oplus \mathcal{O}_{x, x}
\]

The condition \( s_x(\mathcal{F}_x^*) \subset \mathcal{M}_x \) means \( s_x \in \mathcal{M}_x \cdot \mathcal{F}_x \).

If we pass to the quotient by the maximal ideal:

\[
\mathcal{F}_x / \mathcal{M}_x \mathcal{F}_x = \mathcal{O}_{x, x} / \mathcal{M}_x \mathcal{O}_{x, x} = k(x) \oplus \cdots \oplus k(x)
\]

\[
\mathcal{F}_x^* / \mathcal{M}_x \mathcal{F}_x^* \cong (\mathcal{F}_x / \mathcal{M}_x \mathcal{F}_x)^* \cong (k(x) \oplus \cdots)^*
\]
$s_x \in \mathcal{M}_n \Rightarrow \mathcal{M}_n \text{ induces the linear map on } \mathcal{L}(\mathcal{O}(\mathcal{X}))^* \\
\Rightarrow s_x = 0 \in \mathcal{O}(\mathcal{X})_{\mathcal{X}} = \mathcal{O}_x / \mathcal{M}_x \mathcal{O}_x \\
\Rightarrow s_x \in \mathcal{M}_x \mathcal{O}_x \\
\checkmark \\
\text{Claim 2.} \\
\text{Note: this fails if } \mathcal{F} \text{ is not locally free!} \\
\text{Ex: } \mathcal{F} = \text{skyscrapers sheaf supported on a proper closed subset of } X, \text{ e.g. } [2] \\
\text{In this case: } \mathcal{F}^{-1} = 0 \\
\text{Homogeneous ideals of subsets of Projective spaces:} \\
\text{Assume now } S := \mathbb{R}[X_0, \ldots, X_n], \mathbb{R} \text{ rig.} \\
X := \text{Proj } S = \mathbb{P}^n_{\mathbb{R}}.
Then we know, in particular, that

\[ H^0(X, O_X(d)) = S_d \quad \forall \, d \in \mathbb{Z}. \]

**Def.** For a subset \( Y \subset X \), define

\[ I_{Y, d} = \{ s \in S_d \mid Y \subset \mathbb{Z}(s) \} \]

the set of homogeneous polynomials vanishing on \( Y \).

\( \forall x \in m_X \quad \forall \, s(x) = 0 \text{ in } O_{X,m} \)

For a closed subscheme \( Y \subset X \) with ideal sheaf \( I_Y \)

define

\[ I_{Y, d} = \{ s \in S_d \mid I_{Z}(s) \subset I_Y \} . \]

**Ex:** If \( Y \) is reduced, the two \( I_{Y, d} \) are equal.
Def: The homogeneous ideal of a subset or closed subscheme \( Y \subset X \) is:

\[
I_Y := \bigoplus_{d \in \mathbb{Z}} I_{Y,d} \subset S
\]

The homogeneous coordinate ring of \( Y \) is:

\[
S(Y) := S / I_Y
\]