Some properties of separated and proper morphisms:

(All schemes noetherian)

(1) Open and closed embeddings are separated.
Closed embeddings are proper.

(2) Compositions of separated, resp. proper, morphisms are separated, resp. proper.

(3) If $f: X \to Y$ and $g: Y \to Z$ are morphisms, then, if $(g \circ f)$ is separated, then f is separated.

(4) If $f: X \to Y$, $g: Y \to Z$ are morphisms, $g \circ f$ is proper and g is separated, then f is proper.

(5) The property of being separated or proper is local on the base, i.e., $f: X \to Y$ is separated, resp. proper,

iff $\exists$ covering $Y = \bigcup_{i \in I} V_i$

s.t. $\forall\, i$ $f|_{V_i} : f^{-1}(V_i) \to V_i$ is separated,
resp. proper.

(6) separated and proper morphisms are stable under base change, i.e., $\forall$ $f : X \to Y$, $\forall$ $g : Y' \to Y$

form

$$
\begin{tikzcd}
X' = X \times_Y Y' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
Y' \arrow[r] & Y
\end{tikzcd}
$$

f separated $\Rightarrow$ f' separated

f proper $\Rightarrow$ f' proper.

(7) Products of separated, resp. proper, morphisms are separated, resp. proper. i.e., given

$$f : X \longrightarrow Y \,(\longrightarrow S) \qquad g : X' \longrightarrow Y' \,(\longrightarrow S)$$

then f, g separated $\Rightarrow$ $f \times g : X \underset{S}{\times} X' \longrightarrow Y \underset{S}{\times} Y'$ is separated

f, g proper $\Rightarrow$ $f \times g$ proper

<u>Proofs:</u> All properties can be proved using the valuative criteria. We illustrate this by proving (6):

— Suppose given $f : X \to Y$ and a base change $\pi : Y' \to Y$. Assume f is separated. We show $f' = \pi^*(f) : X' \to Y'$ is separated, where $X' := X \underset{Y}{\times} Y'$ and

$$\begin{tikzcd} X' \arrow[r, "\pi'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "\pi"] & Y \end{tikzcd}$$

Given a field K, a valuation ring $R \subset K$ and morphisms

$$\begin{tikzcd} U := \operatorname{Spec} K \arrow[r] \arrow[d] & X' \arrow[d] \\ T := \operatorname{Spec} R \arrow[r] \arrow[ur, shift left, "h_1"] \arrow[ur, shift right, "h_2"'] & Y' \end{tikzcd}$$

where the diagram is commutative,

we show $h_1 = h_2$

Consider

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X' \arrow[r, "\pi'"'] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, shift left, "h_1"] \arrow[ur, shift right, "h_2"'] \arrow[urr, bend left=40, "g_1"] \arrow[urr, bend right=15, "g_2"'] & Y' \arrow[r, "\pi"'] & Y \end{tikzcd}$$

$g_i := \pi' \circ h_i$

By the valuative criterion for the separated morphism f, $g_1 = g_2$. By the universal property of fiber products,

$h_1 = h_2$.

So f' is separated.

Now assume f is proper. Given a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X' \arrow[d] \\ T \arrow[r] & Y' \end{tikzcd}$$

we show there is $h : T \longrightarrow X'$ s.t.

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X' \arrow[d] \\ T \arrow[r] \arrow[ur, "h"] & Y' \end{tikzcd}$$

commutes.

Again consider

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, "h"] \arrow[urr, "g"] & Y' \arrow[r] & Y \end{tikzcd}$$

By the properness of f, $\exists ! \, g$ making the diagram commute. By the universal property of fiber products, $\exists ! \, h$ making the diagram commute.

Finally, note that f' is of finite type:

$$Y = \bigcup_{j \in J} \operatorname{Spec} B_j \qquad f^{-1}(\operatorname{Spec} B_j) = \bigcup_{i=1}^{m_j} \operatorname{Spec} A_{ij}$$

A_{ij} a finitely generated B_j-alg.

$$\pi^{-1}(\operatorname{Spec} B_j) = \bigcup_{k \in K_j} \operatorname{Spec} C_{jk}$$

$$X' = X \times_Y Y' = \bigcup_{i,j,k} \operatorname{Spec}\left(A_{ij} \otimes_{B_j} C_{jk}\right)$$

$$(f')^{-1}(\operatorname{Spec} C_{jk}) = \bigcup_{i=1}^{m_j} \operatorname{Spec}\left(A_{ij} \otimes_{B_j} C_{jk}\right)$$

$A_{ij} \otimes_{B_j} C_{jk}$ is finitely generated over C_{jk}.