In topology, for a continuous map $f: X \to Y$ with X Hausdorff, Y Hausdorff and locally compact (i.e., every point has a compact neighborhood), one can define properness in three equivalent ways:

1) The inverse image of a compact subset is compact.
2) f is closed and the inverse image of every point is compact
3) f is universally closed.

Universally closed means that, for any topological space Z, the map $f \times Id_Z : X \times Z \to Y \times Z$ is closed.

In algebraic geometry, we have an analogous notion of properness for morphisms which, in the case $Y = \operatorname{Spec} k$ replaces the notion of compactness of X.

Since we do not already have a notion of compactness for schemes, we use the concept of universal closedness to define proper morphisms.

This uses base change which is the algebro-geometric generalization of extension of scalars: e.g., given an $\mathbb{R}$ vector space V, we can extend the scalars to $\mathbb{C}$ to obtain the complex vector space $V \otimes_{\mathbb{R}} \mathbb{C}$.

Definition (of base change): Given $X \xrightarrow[\pi_X]{} S$ morphism of schemes, the base change of X to a scheme $S' \xrightarrow[\varphi]{} S$ is the fiber product

$$\begin{tikzcd} X' := X \times_S S' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "\pi_X"] \\ S' \arrow[r, "\varphi"'] & S \end{tikzcd}$$

Definition (of (universally) closed morphisms): A morphism of schemes is closed if the image of any closed subset is closed. A morphism $f: X \to Y$ of schemes is universally closed if for any $Y' \to Y$, the base change $X \times_Y Y' \to Y'$ is closed.

__Example:__ $\mathbb{A}^1 \longrightarrow \operatorname{Spec} k$ is closed, but it is __not__ universally closed: base change to $\mathbb{A}^1 \to \operatorname{Spec} k$:

$$\begin{tikzcd} \mathbb{A}^2_k \cong \mathbb{A}^1 \times_k \mathbb{A}^1 \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

p_2 is not closed: the image of the hyperbola $Z(xy-1) \subset \mathbb{A}^1 \times \mathbb{A}^1$ is not closed in $\mathbb{A}^1$.

__Def:__ A morphism is proper if it is of finite type, separated and universally closed.

__Theorem:__ __The valuative criterion of properness:__

Let $f : X \longrightarrow Y$ be a morphism of finite type, with X noetherian. Then f is proper if and only if,

for any field K and valuation ring $R \subset K$,
and any morphisms $T := \operatorname{Spec} R \to Y$, $U := \operatorname{Spec} K \to X$
forming a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed, "\exists!"'] & Y \end{tikzcd}$$

there is a unique morphism $T \to X$ making the whole diagram above commutative.

<u>Proof:</u> First assume $f : X \to Y$ is proper.
For any data as in the theorem
$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ T \arrow[r] & Y \end{tikzcd}$$
since f is by definition separated,
$\exists$ at most one lift $T \to X$ making the diagram commute

So we need to prove the existence of the lift.

$$\begin{tikzcd}
\operatorname{Spec} K = U \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\operatorname{Spec} R = T \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}$$

Let us do the base change

$$\begin{tikzcd}
X \times_Y T \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
T \arrow[r] \arrow[u, bend left, dotted, "?"] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
U \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_Y T \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
& T \arrow[r] & Y
\end{tikzcd}$$

$U \to X \times_Y T$ exists and is unique by the universal property of fiber products.

Let $\xi_1 \in X \times_Y T$ be the image of V, let

$Z := \overline{\{\xi_1\}}$ be the closure of ξ_1 in $X \times_Y T$ with the reduced induced scheme structure.

f universally closed $\Rightarrow$ p_2 is closed $\Rightarrow$ $p_2(Z) \subset T$ is closed

$p_2(Z)$ contains $p_2(\xi_1)$ which is the generic point of T

$\Rightarrow$ $p_2(Z) = T$

$\Rightarrow$ $\exists$ $\xi_0 \in Z$ s.t. $p_2(\xi_0) =$ the closed point of T.

$$p_2 : Z \longrightarrow T, \quad \xi_1 \longmapsto t_1, \quad \xi_0 \longmapsto t_0$$

$\Rightarrow$ local hom. of local rings

$$\begin{tikzcd} R = \mathcal{O}_{T,t_0} \arrow[r] \arrow[d, hook] & \mathcal{O}_{Z,\xi_0} \arrow[d, hook] \\ K = \mathcal{O}_{T,\xi_1} \arrow[r] & \mathcal{O}_{Z,\xi_1} = K \end{tikzcd}$$

$\Rightarrow \quad \mathcal{O}_{Z,\xi_0}$ dominates R

$\Rightarrow \quad \mathcal{O}_{Z,\xi_0} = R$ because R is a valuation ring.
(maximal for dominance)

By the lemma (4.4 in Hartshorne) from one or two lectures ago, $\exists$ morphism $T \longrightarrow X \times_Y T$ sending t_0 to ξ_0 and t_1 to ξ_1.

Now compose with p_1 to obtain the desired lift

$$T \longrightarrow X.$$

Conversely, suppose f is of finite type, X noetherian and the valuative criterion holds.

By the valuative criterion of separatedness, we know f is separated.

We need to prove that f is universally closed.

For any $Y' \to Y$, we show that $X \times Y' \xrightarrow[p_2]{} Y'$ is closed.

$$\begin{tikzcd} X' := X \times_Y Y' \arrow[r, "p_1"] \arrow[d, "f' := p_2"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

Let $Z \subset X'$ be a closed subset. $f'(Z) \subset Y'$

Endow Z with the reduced induced scheme structure.

exercise: f' is of finite type, and $f'|_Z$ is of finite type.

$\Rightarrow f'|_Z$ is quasi-compact.

We use a lemma (4.5 in Hartshorne) from previous lectures:

$f'(Z)$ is closed iff it is closed under specialization.

Let $z_1 \in Z$, put $y_1 := f(z_1)$

For any $y_0 \in \overline{\{y_1\}}$ a specialization, we show $y_0 \in f^{-1}(Z)$.

Endow $W := \overline{\{y_1\}}$ with its reduced induced scheme structure.

$$\mathcal{O}_{W,y_0} \hookrightarrow \mathcal{O}_{W,y_1} = K_W \hookrightarrow K := k(z_1)$$

residue field in Z

Let R be a valuation ring of K dominating $\mathcal{O}_{W,y_0}$.

Apply Lemma 4.4 again to obtain a morphism

$$\begin{array}{ccccc} U = \operatorname{Spec} K \hookrightarrow & T = \operatorname{Spec} R & \longrightarrow & Y' & \\ \longmapsto & t_1 & \longmapsto & y_1 & = f(z_1) \\ & t_0 & \longmapsto & y_0 & \end{array}$$

By construction, we have the commutative diagram

$$\begin{tikzcd}
t_1 \arrow[r, mapsto] & z_1 & & \\
U = \operatorname{Spec} K \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & Z \arrow[d, "f'|_Z"] & \ni z_1 \\
T = \operatorname{Spec} R \arrow[r] & Y' & \\
t_0 \arrow[r, mapsto] & y_0 & \\
t_1 \arrow[r, mapsto] & y_1 &
\end{tikzcd}$$

Compose:

$$\begin{tikzcd}
t_1 \in U \arrow[r] \arrow[d] \arrow[rrr, bend left] & Z \arrow[r, hook] \arrow[dr, "f'|_Z"] & X' \arrow[r] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
t_0, t_1 \in T \arrow[rr] \arrow[urr, "\circlearrowleft"] \arrow[urrr] \arrow[rrr, bend right] & & Y' \arrow[r] & Y
\end{tikzcd}$$

By the valuative criterion, $\exists !$ lift $T \longrightarrow X$ making the diagram commutative. $\Rightarrow$ $T \longrightarrow X'$ by the universal property of fiber products.

The generic point of T maps into Z, and Z is closed, hence all of T maps into Z.

Let $z_0 \in Z$ be the image of $t_0 \in T$.

Then $f'(z_0) = \text{image of } t_0 = y_0 \in f'(Z)$.

$\square$