Some terminology: When $x_0 \in \overline{\{x_1\}}$, we say

- $x_0$ is a specialization of $x_1$
- $x_1$ is a generization of $x_0$

Proof of the valuative criterion of separatedness:

First suppose $f: X \to Y$ is separated.

Suppose we have
$$\begin{tikzcd}
\operatorname{Spec} K = U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec} R = T \arrow[r] \arrow[ur, dashed, shift left, "h"] \arrow[ur, dashed, shift right, "h'"'] & Y
\end{tikzcd}$$
and two morphisms $h, h'$ making the diagram commutative. We show $h = h'$.

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[rrrd, bend left, "h|_U"] \arrow[rrd, "(h,h')|_U"'] \arrow[rrdd, bend right, "h'|_U"'] & T \arrow[rrd, bend left, "h"] \arrow[rd, "(h,h')"] \arrow[rdd, bend right, "h'"'] & & \\
& & X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
& & X \arrow[r, "f"] & Y
\end{tikzcd}$$

$h|_U = h'|_U$

<u>Claim</u>: (follows from the universal property of fiber products) exercise.

a morphism $g: W \to X \times_Y X$ $W$ any scheme
factors through $\Delta: X \to X \times_Y X$ iff $p_1 \circ g = p_2 \circ g$

$h|_U = h'|_U \Rightarrow$ $(h,h')|_U : U \to X \times_Y X$ factors

through $\Delta$: $(h,h')|_U : U \to X \xrightarrow{\Delta} X \times_Y X$

$$\begin{tikzcd} U \arrow[d, hook] \\ T \end{tikzcd}$$

image is the generic point of $T$

$\Delta$ is a closed embedding and $\Delta(X)$ contains the image of the generic point of $T \Rightarrow \Delta(X) \supset$ image of $T$

$\Rightarrow \quad \Delta(x) \ni$ image of the closed point of $T = \operatorname{Spec} R$

Claim (from previous page) $\Rightarrow$ $h(t_0) = h'(t_0) =: x_0$

$x_1 := h(t_1) = h'(t_1)$ = image of $U = \operatorname{Spec} K$

$\Rightarrow \quad x_0 \in \overline{\{x_1\}} =: Z$ with reduced induced scheme structure

and $\quad k(x_1) = \mathcal{O}_{Z,x_1} \subset K$ given by $h|_U = h'|_U$

$U = \operatorname{Spec} K \longrightarrow X$

$x_0 = h(t_0) = h'(t_0) \Rightarrow \mathcal{O}_{Z,x_0} \subset R$ dominated.

previous lemma $\Rightarrow$ $h = h' : \operatorname{Spec} R \longrightarrow X$

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Conversely, suppose the valuative criterion is satisfied.

We show that $\Delta(X) \subset X \times_Y X$ is closed.

We use the following lemma:

**Lemma:** Let $f: X \to Y$ be a quasi-compact morphism of schemes. The subset $f(X)$ of $Y$ is closed iff it is closed under specialization, meaning $\forall\, y \in f(X)$, any specialization of $y$ also belongs to $f(X)$.

**Proof:** Hartshorne

**Remark:** In general any closed subset is closed under specialization.

---

Since $X$ is noetherian, $\Delta: X \longrightarrow X \times_Y X$ is quasi-compact.

We show $\Delta(X)$ is closed under specialization.

Choose $\xi_1 \in \Delta(X) \subset X \times_Y X$ and $\xi_0 \in Z := \overline{\{\xi_1\}} \subset X \times_Y X$

we show $\xi_0 \in \Delta(X)$.

endow $Z$ with the reduced induced scheme structure. $\xi_1$ is the generic point of $Z$

$$\Rightarrow \quad \mathcal{O}_{Z,\xi_1} =: K \text{ is a field}$$

$= k(\xi_1)$ residue field of $\xi_1$ in $X \times_Y X$

$$\mathcal{O}_{Z,\xi_0} \subset K$$

$\exists$ valuation ring $R \subset K$ which dominates $\mathcal{O}_{Z,\xi_0}$.

Using the lemma from the previous lecture, we obtain a morphism $g : T := \operatorname{Spec} R \longrightarrow X \times_Y X$

s.t. $g(t_1) = \xi_1$, $g(t_0) = \xi_0$

$h := p_1 \circ g \qquad h' := p_2 \circ g \qquad h|_U = h'|_U \qquad U = \operatorname{Spec} K$

because $\xi_1 = g(t_1) \in \Delta(X)$

(see Claim about $\Delta$)

So $(p_1 \circ g)|_U = (p_2 \circ g)|_U \xRightarrow{\text{valuative criterion}} p_1 \circ g = p_2 \circ g$

(use claim $\Delta$ again) $\Rightarrow$ $g$ factors through $\Delta$.

$\Rightarrow \xi_0 = g(t_0) \in \Delta(X)$. $\square$

Recall that properness is a substitute for compactness.
In a compact top. space, any closed subset is also compact, so any continuous map from a compact topological space is closed.
We use the condition of universally closed to define properness: being closed is not enough: $\mathbb{A}^1 \longrightarrow \operatorname{Spec} k$ is closed but it is not proper.

$$\begin{array}{c} \mathbb{A}^1 \\ \cap \\ \mathbb{P}^1 \end{array}$$