

Definition: The dimension of a scheme is its dimension as a topological space (i.e., the maximum length of a chain of irreducible closed subsets).

For any irreducible closed subset Z of X , the codimension of Z in X is the supremum of the set of integers n s.t. \exists a chain of irreducible closed subsets $Z = Z_0 \subsetneq Z_1 \subsetneq Z_2 \subseteq \dots \subseteq Z_n \subset X$.

Caution: For varieties over a field,
$$\dim Z + \operatorname{codim}_X Z = \dim X$$

(follows from $\dim A = \dim A_p + \operatorname{height} p$
A finitely gen. alg. / field.)

However, this is NOT true for general schemes.

(see II.3.2.8, Ex. II.3.20, 21, 22)

For affine schemes: $\dim \operatorname{Spec} A = \dim A$
(Kull dim.)

Question: Can we define the fibers of a morphism of schemes, as schemes?

YES: fiber products.

We can in fact define the inverse image of any subscheme of Y by a morphism $X \rightarrow Y$.

Def: Given two morphisms

$$\pi_X: X \longrightarrow S, \quad \pi_Y: Y \longrightarrow S$$

the fiber product of X and Y over S is the scheme,

denoted $X \times_S Y$ s.t. \exists a commutative diagram

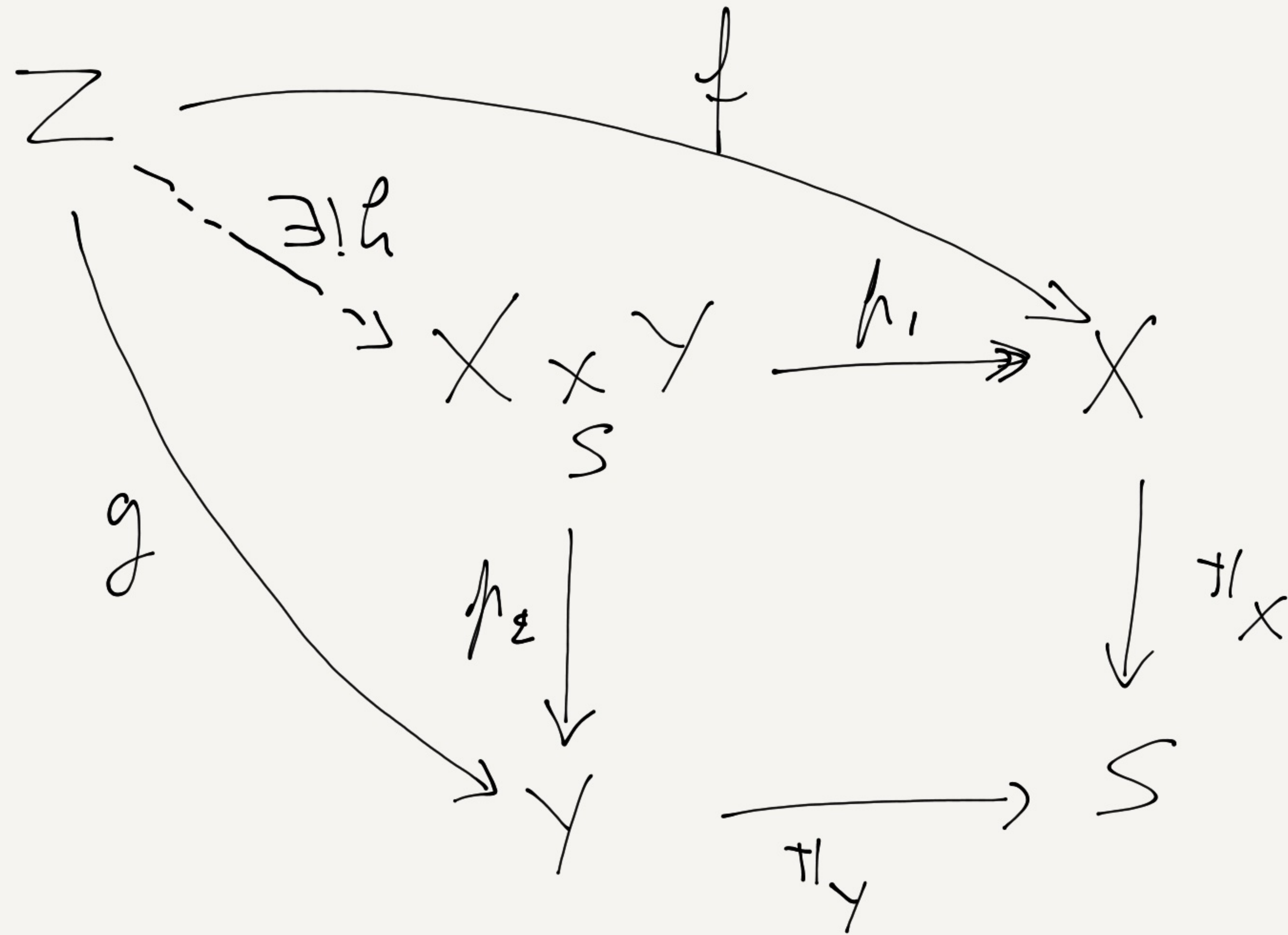
$$\begin{array}{ccc} X \times_S Y & \xrightarrow{h_1} & X \\ \downarrow h_2 & \circlearrowleft & \downarrow \pi_X \\ Y & \xrightarrow{\pi_Y} & S \end{array}$$

s.t. \forall schemes Z and morphisms $f: Z \rightarrow X$,

$$g: Z \rightarrow Y \text{ with } \pi_X \circ f = \pi_Y \circ g =: \pi_Z$$

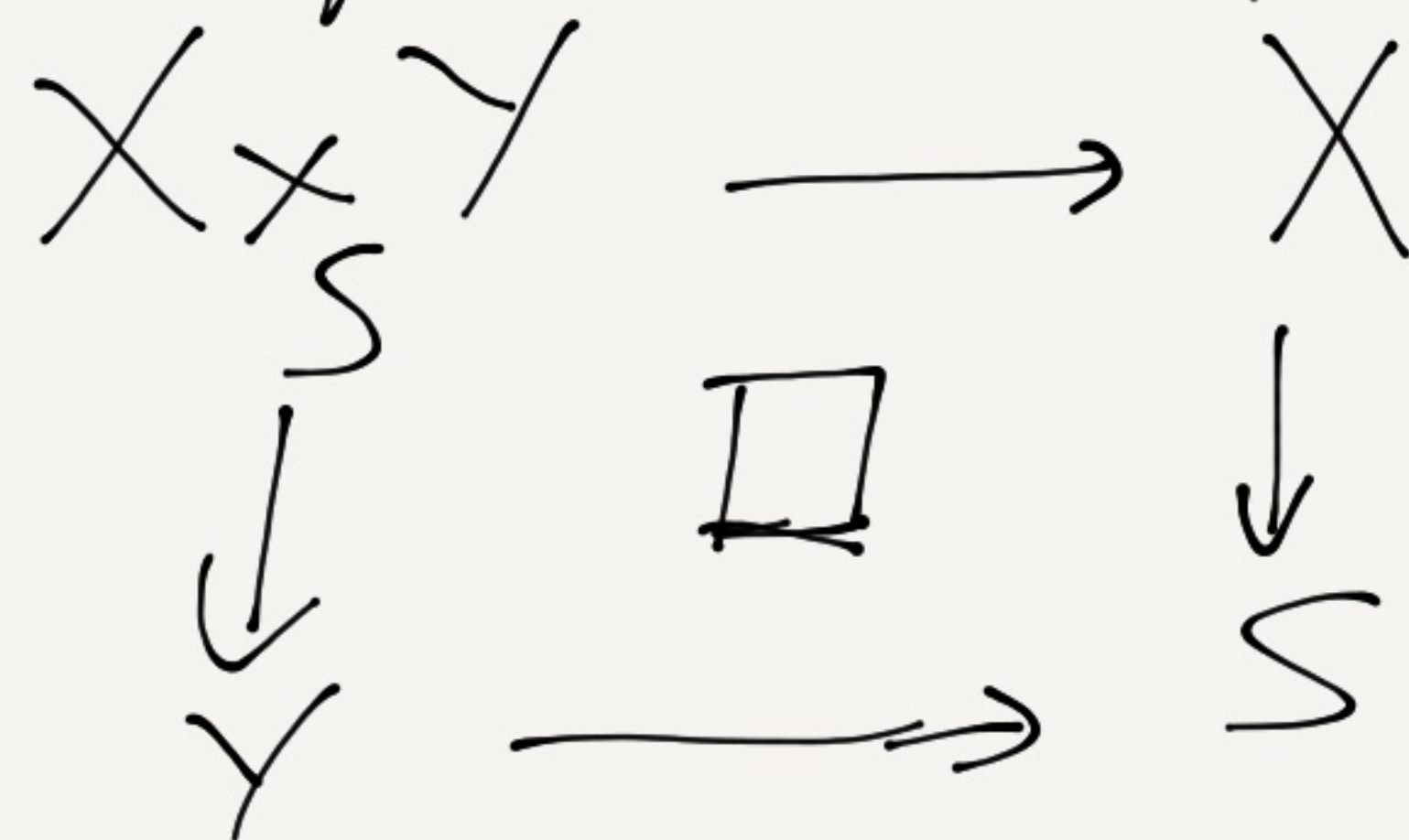
$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ g \downarrow & \circlearrowleft & \downarrow \pi_X \\ Y & \xrightarrow{\pi_Y} & S \end{array}$$

$\exists!$ $h: Z \rightarrow X \times_S Y$ s.t. $p_1 h = f, p_2 h = g$:



The universal property implies that, $X \times_S Y$, if it exists, is unique up to unique isomorphism.

Notation:



this is often called a Cartesian diagram.

Existence of fiber products:

We first construct fiber products of affine schemes, then we glue.

$$S = \text{Spec } R, \quad X = \text{Spec } A, \quad Y = \text{Spec } B$$

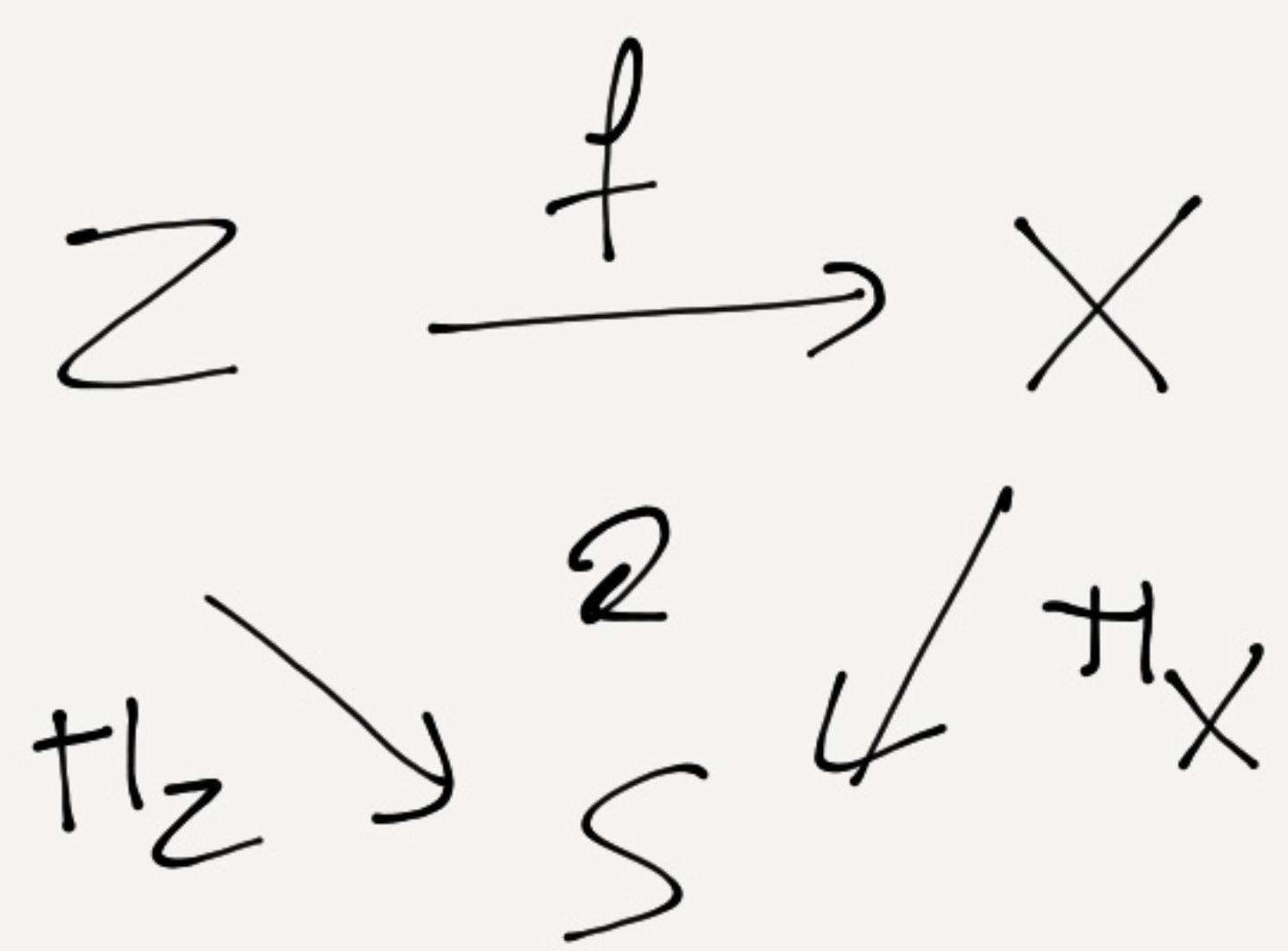
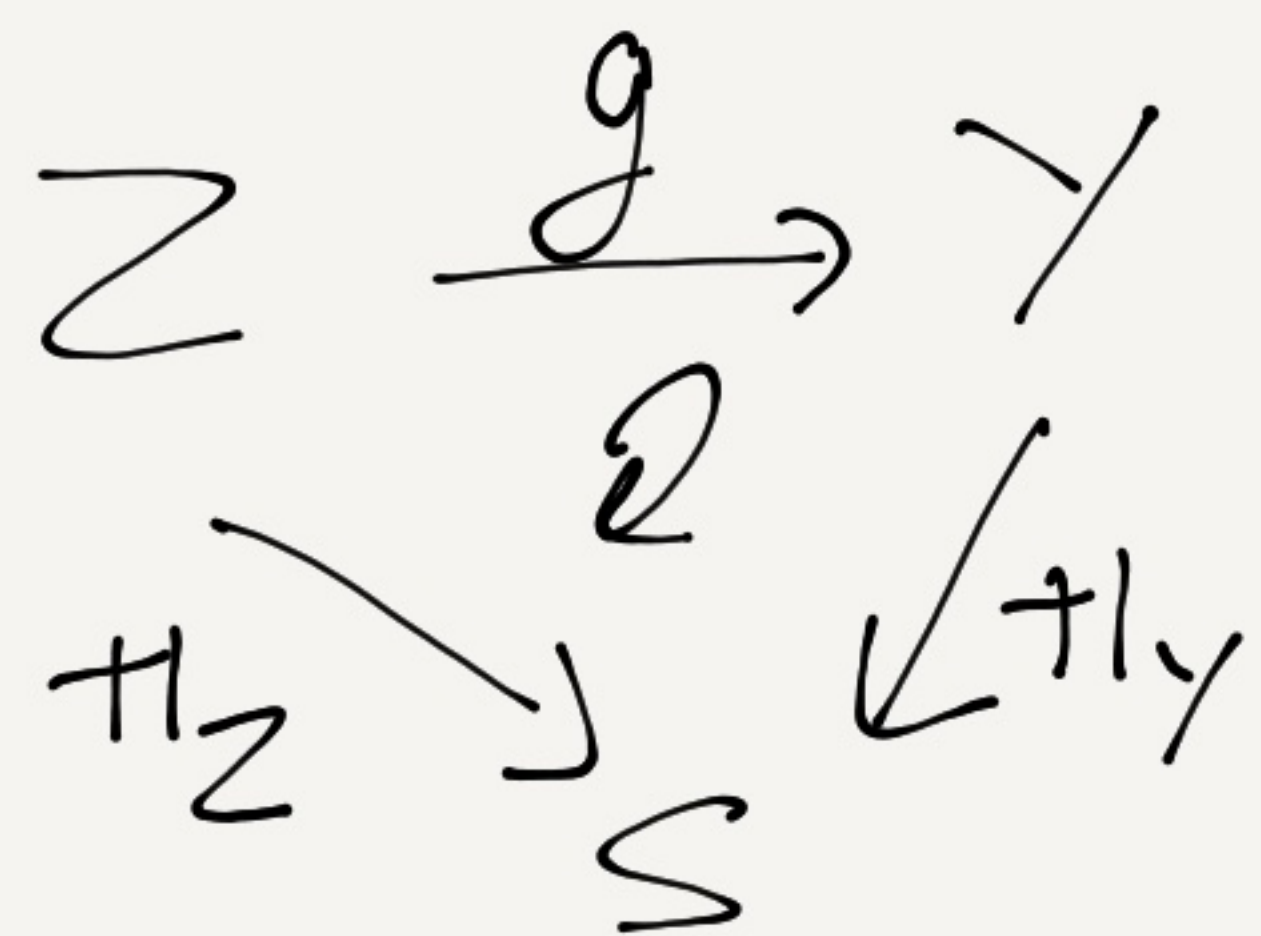
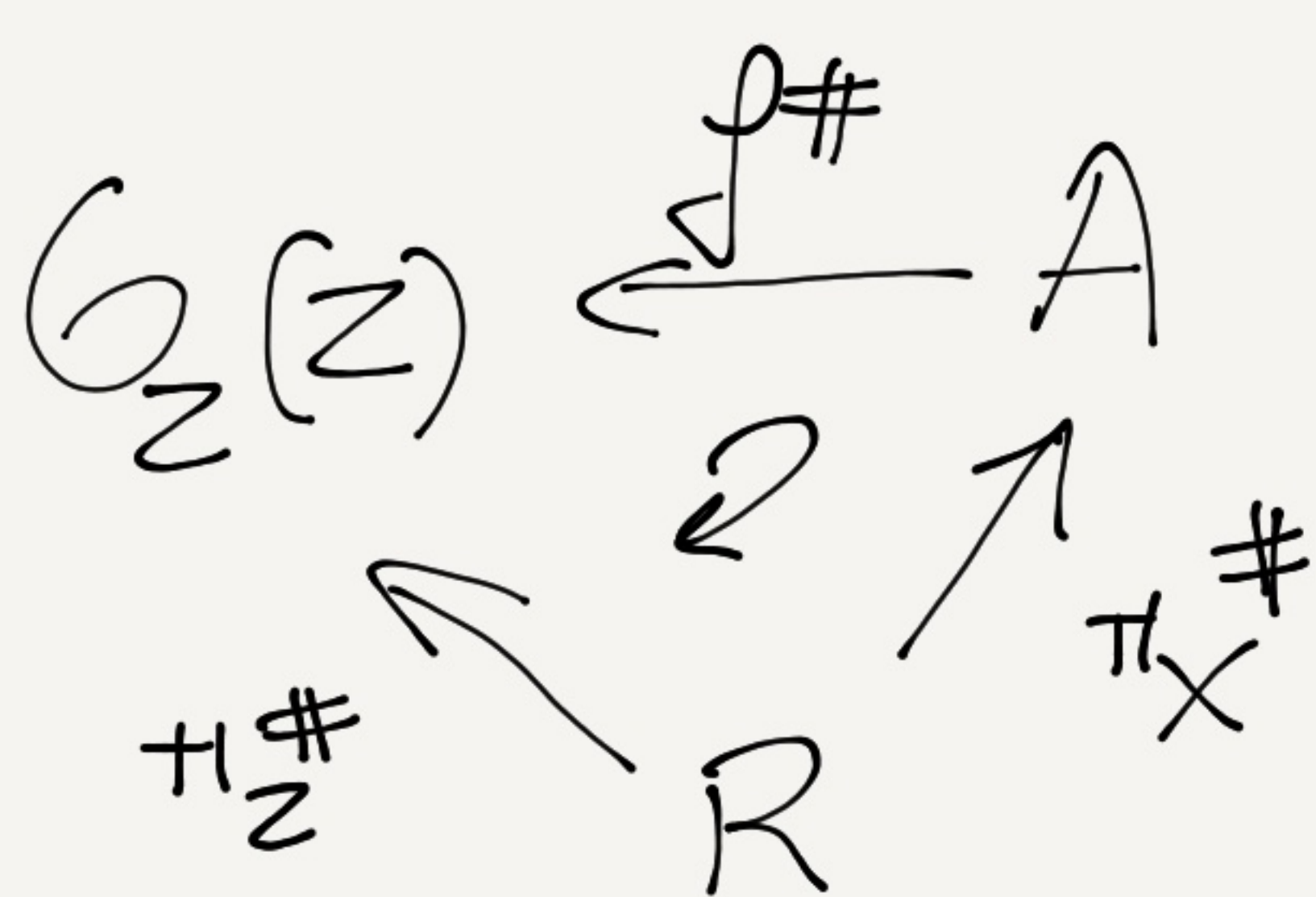
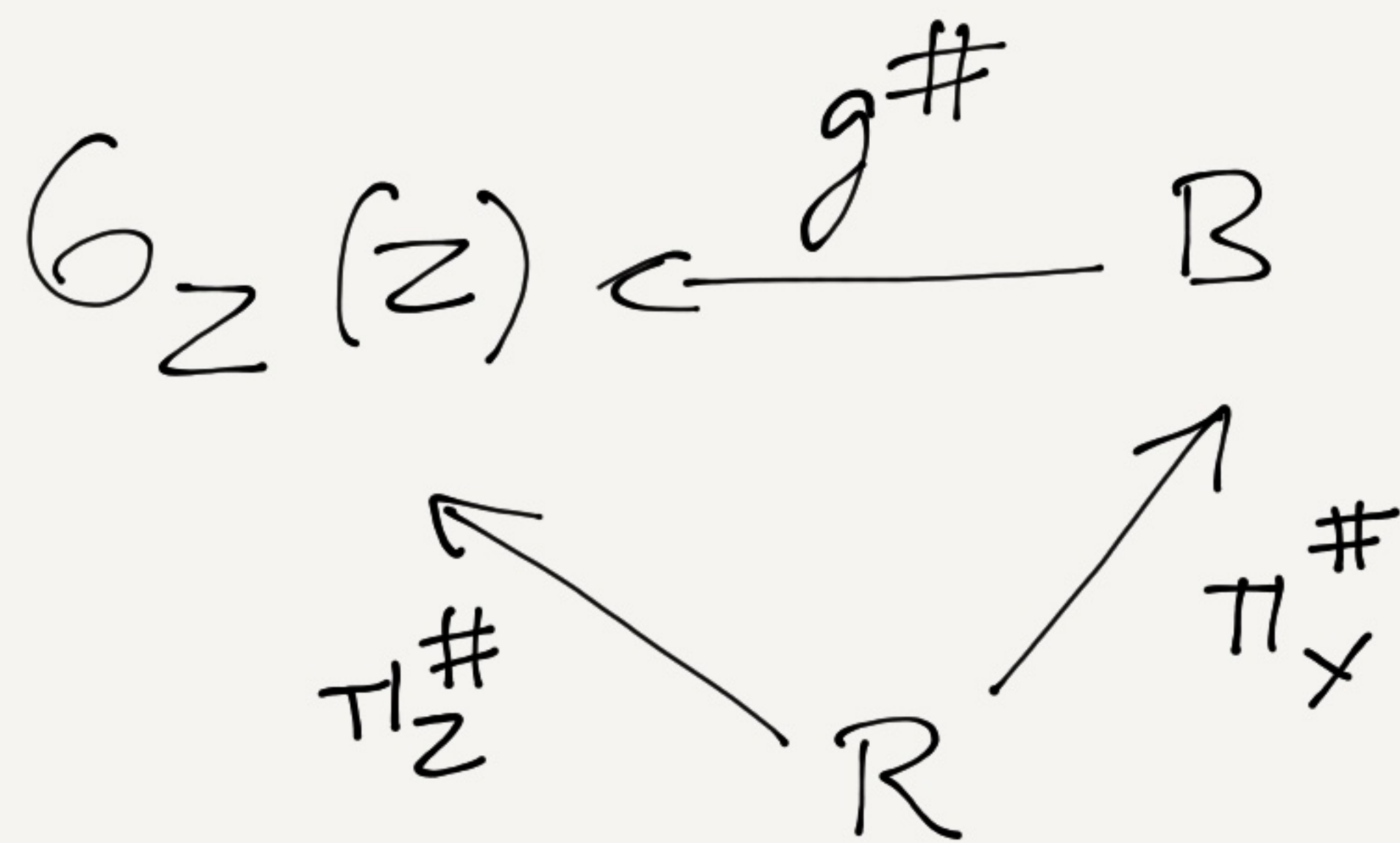
$$\begin{array}{ccc}
 & X & \\
 & \downarrow \pi_X & \\
 Y & \longrightarrow & S \\
 & \pi_Y &
 \end{array}
 \iff
 \begin{array}{ccc}
 & A & \\
 & \uparrow & \\
 B & \longleftarrow & R
 \end{array}$$

Define $X \times_S Y := \text{Spec } A \otimes_R B$

$$\begin{array}{ccccc}
 & & a \otimes 1 & \longleftarrow & a \\
 & & A \otimes B & \longleftarrow & A \\
 \text{loc} & \uparrow & \uparrow & & \uparrow \\
 & & R & & R \\
 & & \downarrow & & \downarrow \\
 & & B & \longleftarrow & R
 \end{array}$$

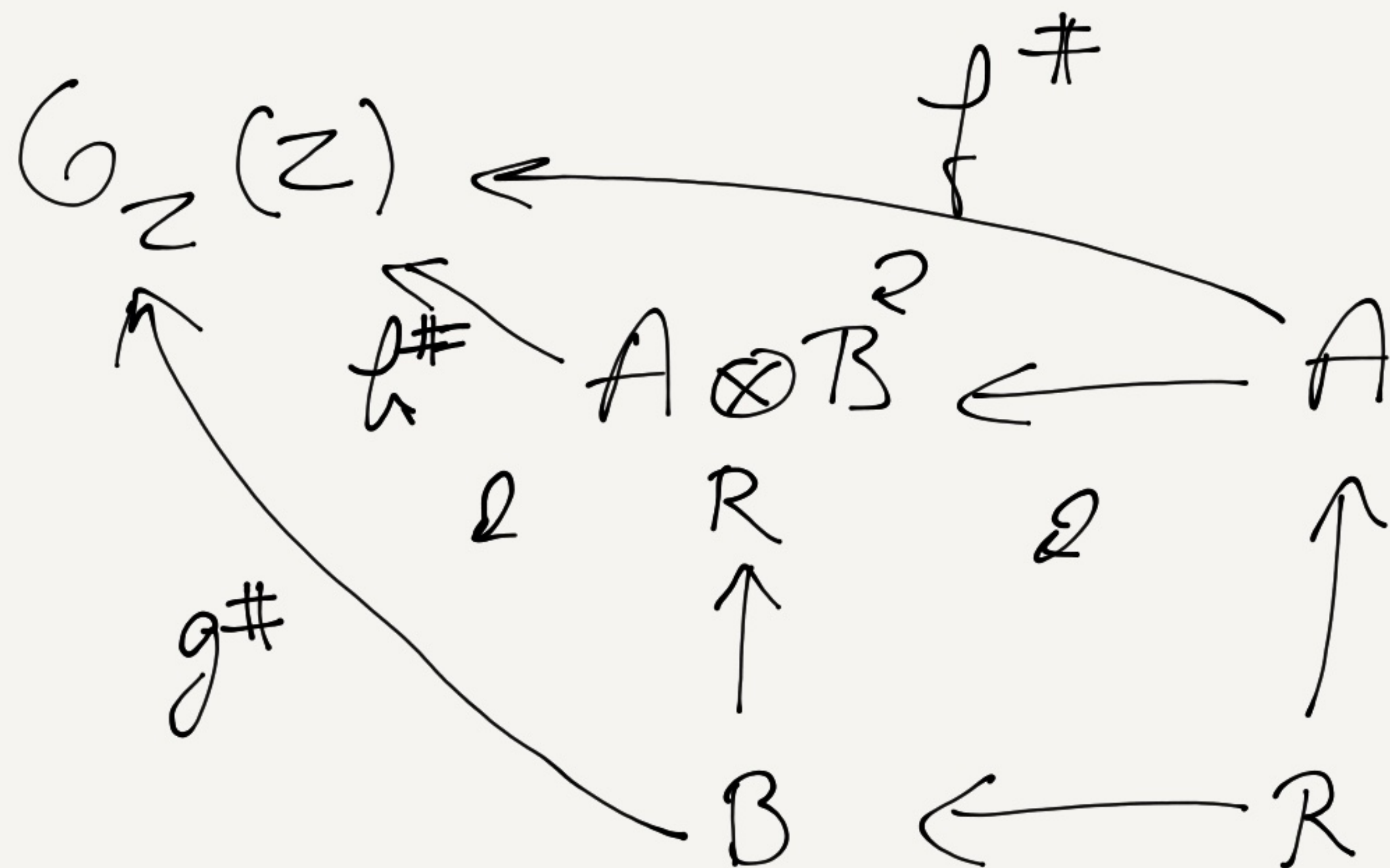
We verify the universal property:

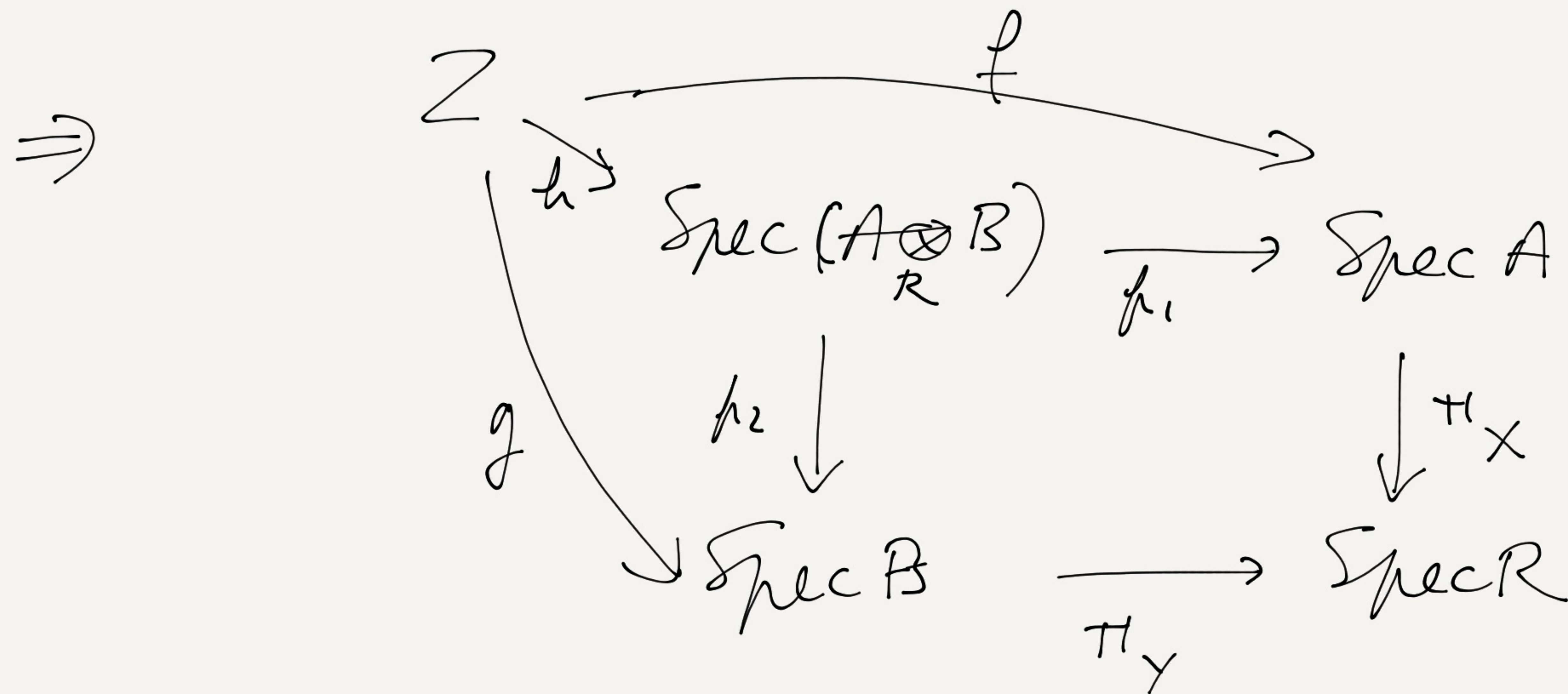
$$\forall \begin{array}{ccc}
 Z & \xrightarrow{f} & X \\
 \pi_Z \searrow & & \swarrow \pi_X \\
 & S &
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 Z & \xrightarrow{g} & Y \\
 \pi_Z \searrow & & \swarrow \pi_Y \\
 & S &
 \end{array}$$


 (\Leftrightarrow)

 (\Leftrightarrow)

 \Rightarrow

$$\begin{array}{ccc}
 A \otimes B & \xrightarrow{h^\#} & \mathcal{G}_Z(Z) \\
 R & & \\
 a \otimes b & \mapsto & f^\#(a) \cdot g^\#(b)
 \end{array}$$

(exercise)

 \Rightarrow




Note: The underlying topological space of $X \times_S Y$ as defined above is NOT equal to the product of the topological spaces of $\text{Spec } A$ and $\text{Spec } B$.

In fact, even the underlying sets are different:

Ex. II.3.9 in homework 9. (alg. closed)

However, in the case of varieties over a field, the set of closed points of the fiber product is the product of the

sets of closed points of the factors. (Ex. II.3.23)

Also see homework 1.

To construct general fiber products, we glue: (detail in Hartshorne):

Arbitrary schemes: $S = \bigcup_{i \in I} U_i$ $U_i = \text{Spec } R_i$

$X \xrightarrow{\pi_X} S$ $\pi_X^{-1}(U_i) = \bigcup_{j \in J_i} V_{ij}$ $V_{ij} = \text{Spec } A_{ij} \subset X$

$\pi_Y^{-1}(U_i) = \bigcup_{k \in K_i} W_{ik}$ $W_{ik} = \text{Spec } B_{ik} \subset Y$

$\text{Spec } A_{ij} \downarrow$
 $\text{Spec } B_{ik} \rightarrow \text{Spec } A_i$

for u $V_{ij} \times_{U_i} W_{ik} := \text{Spec } A_{ij} \otimes_{R_i} B_{ik}$

show that these glue to give $X \times_S Y$. \square

Examples next time.

Fibers of a morphism:

X a scheme, $x \in X$ $\mathcal{O}_{X,x} \supset m_x$

Def: The residue field of X at x is

$$k(x) := \mathcal{O}_{X,x} / m_x$$

Ex. II.2.7 The datum of a morphism from $\text{Spec } K$ (K any field) to X with image x is equivalent to the datum of an inclusion $k(x) \hookrightarrow K$.

\Rightarrow Given $x \in X$, we always have a morphism $\text{Spec } k(x) \rightarrow X$ with image x , using the identity $k(x) \xrightarrow{\text{id}} k(x)$.

Def: Given a morphism of schemes $f: X \rightarrow Y$, let $y \in Y$ be a point, and $\text{Spec } k(y) \hookrightarrow Y$ be the natural morphism as above.

The fiber X_y of f at y is

$$X_y := X \times_Y \text{Spec } k(y) \text{ i.e.,}$$

$$\begin{array}{ccc} X_y & \longrightarrow & X \\ \downarrow & \square & \downarrow f \\ \text{Spec } k(y) & \longrightarrow & Y \end{array}$$

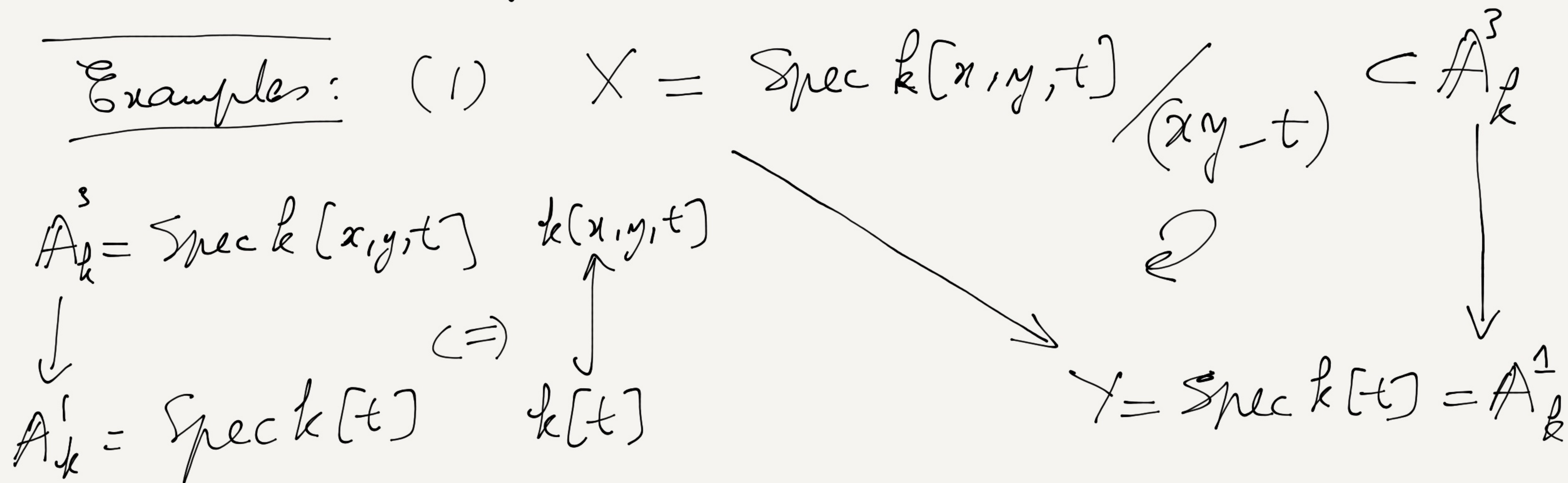
Ex. II.3.10: The underlying topological space of X_y

$$\text{is } f^{-1}(y) \subset X.$$

This allows us to think of $f: X \rightarrow Y$ as a family of

schemes $\{X_y, y \in Y\}$ parametrized by Y .

Def: Given a field k , a scheme X_0/k ,
 (i.e., $X_0 \rightarrow \text{Spec } k$) a deformation of X_0 over a
 scheme Y over k is a k -morphism $X \rightarrow Y$ (i.e., $X \rightarrow Y$
 $\downarrow \text{Spec } k$)
 s.t. $\exists y_0 \in Y$ with $k(y_0) \cong k$ and $X_{y_0} \cong X_0$ as k -schemes.
 The other fibers of f (at k -rational points, i.e., y s.t. $k(y) \cong k$)
 are called deformations of X_0 .



Some fibres! $t = a \in k \rightsquigarrow y \in Y$
 $\rightsquigarrow (t-a) \subset k[t]$

$$X_y := \text{Spec } k[x, y, t] / (xy-t) \times_Y \text{Spec } k(y)$$

$$= \text{Spec} \left(k[x, y, t] / (xy-t) \otimes_{k[t]} k(y) \right)$$

$$k(y) = \mathcal{O}_{Y, y} / \mathfrak{m}_y = k[t]_{(t-a)} / (t-a)_{(t-a)} \cong k.$$

$$0 \rightarrow (t-a) \rightarrow k[t] \rightarrow k[t] / (t-a) = k \rightarrow 0$$

localize!

$$0 \rightarrow (t-a)_{(t-a)} \rightarrow k[t]_{(t-a)} \rightarrow k \rightarrow 0$$