<u>Definition:</u> The dimension of a scheme is its dimension as a topological space. (i.e., the maximum length of a chain of irreducible closed subsets).

For any irreducible closed subset Z of X, the codimension of Z in X is the supremum of the set of integers n s.t. $\exists$ a chain of irreducible closed subsets $Z = Z_0 \subsetneq Z_1 \subsetneq Z_2 \subseteq \ldots\ldots \subseteq Z_n \subset X$.

<u>Caution:</u> For varieties over a field,

$$\dim Z + \operatorname{codim}_X Z = \dim X$$

(follows from $\dim A = \dim A/\mathfrak{p} + \operatorname{height} \mathfrak{p}$
A finitely gen. alg. / field.)

However, this is NOT true for general schemes.
(see II.3.2.8, Ex. II.3.20, 21, 22)
For affine schemes: dim Spec A = dim A
(Krull dim.)

__Question:__ Can we define the fibers of a morphism of schemes, as schemes?

__YES__: fiber products.

We can in fact define the inverse image of any subscheme of Y by a morphism $X \longrightarrow Y$.

Def: Given two morphisms

$$\pi_X : X \longrightarrow S \quad , \quad \pi_Y : Y \longrightarrow S$$

the fiber product of X and Y over S is the scheme, denoted $X \times_S Y$ s.t. $\exists$ a commutative diagram

$$\begin{tikzcd}
X \times_S Y \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\pi_X"] \\
Y \arrow[r, "\pi_Y"'] & S
\end{tikzcd}$$

s.t. $\forall$ schemes Z and morphisms $f : Z \to X$,
$g : Z \to Y$ with $\pi_X \circ f = \pi_Y \circ g =: \pi_Z$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\pi_X"] \\
Y \arrow[r, "\pi_Y"'] & S
\end{tikzcd}$$

$\exists!\ h : Z \longrightarrow X \times_S Y$ s.t. $p_1 h = f,\ p_2 h = g$;

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\exists! h"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& X \times_S Y \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "\pi_X"] \\
& Y \arrow[r, "\pi_Y"'] & S
\end{tikzcd}$$

The universal property implies that, $X \times_S Y$, if it exists, is unique up to unique isomorphism.

<u>Notation:</u>

$$\begin{tikzcd}
X \times_S Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\
Y \arrow[r] & S
\end{tikzcd}$$

this is often called a Cartesian diagram.

Existence of fiber products:

We first construct fiber products of affine schemes, then we glue.

$$S = \operatorname{Spec} R \ , \quad X = \operatorname{Spec} A \ , \quad Y = \operatorname{Spec} B$$

$$\begin{tikzcd} & X \arrow[d, "\pi_X"] \\ Y \arrow[r, "\pi_Y"'] & S \end{tikzcd} \iff \begin{tikzcd} & A \\ B & R \arrow[l] \arrow[u] \end{tikzcd}$$

Define $X \times_S Y := \operatorname{Spec} A \otimes_R B$

$$\begin{tikzcd} a\otimes 1 & a \arrow[l, mapsto] \\ A \otimes_R B & A \arrow[l] \\ B \arrow[u] & R \arrow[l] \arrow[u] \end{tikzcd} \qquad 1\otimes b \mapsfrom b$$

We verify the universal property:

$$\forall \quad \begin{tikzcd} Z \arrow[r, "f"] \arrow[dr, "\pi_Z"'] & X \arrow[d, "\pi_X"] \\ & S \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Z \arrow[r, "g"] \arrow[dr, "\pi_Z"'] & Y \arrow[d, "\pi_Y"] \\ & S \end{tikzcd}$$

$$\begin{tikzcd} Z \arrow[rr, "f"] \arrow[dr, "\pi_Z"'] & & X \arrow[dl, "\pi_X"] \\ & S & \end{tikzcd} \iff \begin{tikzcd} \mathcal{O}_Z(Z) & & A \arrow[ll, "f^\#"'] \\ & R \arrow[ul, "\pi_Z^\#"] \arrow[ur, "\pi_X^\#"'] & \end{tikzcd}$$

$$\begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, "\pi_Z"'] & & Y \arrow[dl, "\pi_Y"] \\ & S & \end{tikzcd} \iff \begin{tikzcd} \mathcal{O}_Z(Z) & & B \arrow[ll, "g^\#"'] \\ & R \arrow[ul, "\pi_Z^\#"] \arrow[ur, "\pi_X^\#"'] & \end{tikzcd}$$

$$\Rightarrow \quad A \otimes_R B \xrightarrow{h^\#} \mathcal{O}_Z(Z)$$

$$a \otimes b \longmapsto f^\#(a) \cdot g^\#(b)$$

(exercise)

$$\Rightarrow \quad \begin{tikzcd} \mathcal{O}_Z(Z) & A \otimes_R B \arrow[l, "h^\#"'] & A \arrow[l] \arrow[ll, bend right, "f^\#"'] \\ & B \arrow[u] \arrow[ul, bend left, "g^\#"] & R \arrow[l] \arrow[u] \end{tikzcd}$$

$$\Rightarrow \quad \begin{tikzcd}
Z \arrow[dr, "h"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& \mathrm{Spec}(A \otimes_R B) \arrow[r, "p_1"] \arrow[d, "p_2"'] & \mathrm{Spec}\, A \arrow[d, "\pi_X"] \\
& \mathrm{Spec}\, B \arrow[r, "\pi_Y"'] & \mathrm{Spec}\, R
\end{tikzcd}$$

<u>Note</u>: The underlying topological space of $X \times_S Y$ as defined above is NOT equal to the product of the topological spaces of Spec A and Spec B.

In fact, even the underlying sets are different:

Ex. II.3.9 in homework 9.

However, in the case of varieties over a (alg. closed) field, the set of closed points of the fiber product is the product of the sets of closed points of the factors. (Ex. II.3.23)

Also see homework 1.

To construct general fiber products, we glue: (details in Hartshorne):

Arbitrary schemes: $S = \bigcup_{i \in I} U_i \qquad U_i = \operatorname{Spec} R_i$

$X \xrightarrow{\pi_X} S \qquad \pi_X^{-1}(U_i) = \bigcup_{j \in J_i} V_{ij} \qquad V_{ij} = \operatorname{Spec} A_{ij} \subset X$

$\pi_Y^{-1}(U_i) = \bigcup_{k \in K_i} W_{ik} \qquad W_{ik} = \operatorname{Spec} B_{ik} \subset Y$

$$\begin{array}{ccc} & & \operatorname{Spec} A_{ij} \\ & & \downarrow \\ \operatorname{Spec} B_{ik} & \longrightarrow & \operatorname{Spec} A_i \end{array}$$

form $V_{ij} \times_{U_i} W_{ik} := \operatorname{Spec} A_{ij} \otimes_{R_i} B_{ik}$

show that these glue to give $X \times_S Y$. $\square$

Examples next time.

Fibers of a morphism:

X a scheme, $x \in X$ $\quad \mathcal{O}_{X,x} \supset \mathfrak{m}_x$

Def: The residue field of X at x is

$$k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$$

Ex. II.2.7 The datum of a morphism from Spec K (K any field) to X with image x is equivalent to the datum of an inclusion $k(x) \hookrightarrow K$.

$\Rightarrow$ Given $x \in X$, we always have a morphism $\mathrm{Spec}\, k(x) \to X$ with image x, using the identity $k(x) \xrightarrow{=} k(x)$.

Def: Given a morphism of schemes $f: X \to Y$, let $y \in Y$ be a point, and $\mathrm{Spec}\, k(y) \hookrightarrow Y$ be the natural morphism as above.

The fiber X_y of f at y is

$$X_y := X \times_Y \operatorname{Spec} k(y), \text{ i.e.,}$$

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ \operatorname{Spec} k(y) \arrow[r, hook] & Y \end{tikzcd}$$

Ex. II.3.10: The underlying topological space of X_y is $f^{-1}(y) \subset X$.

This allows us to think of $f : X \to Y$ as a family of schemes $\{X_y, y \in Y\}$ parametrized by Y.

Def: Given a field k, a scheme X_0/k, (i.e., $X_0 \longrightarrow \operatorname{Spec} k$) a deformation of X_0 over a scheme Y over k is a k-morphism $X \longrightarrow Y$ (i.e.,
$$\begin{tikzcd}[column sep=small] X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd}$$
)

s.t. $\exists\, y_0 \in Y$ with $k(y_0) \cong k$ and $X_{y_0} \cong X_0$ as k-schemes.

The other fibers of f (at k-rational points, i.e., y s.t. $k(y) \cong k$) are called deformations of X_0.

Examples: (1)
$$\begin{tikzcd} X = \operatorname{Spec} k[x,y,t]/(xy-t) \arrow[dr] \arrow[r, phantom, "\subset"] & \mathbb{A}^3_k \arrow[d] \\ & Y = \operatorname{Spec} k[t] = \mathbb{A}^1_k \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{A}^3_k = \operatorname{Spec} k[x,y,t] \arrow[d] & & k[x,y,t] \\ \mathbb{A}^1_k = \operatorname{Spec} k[t] & \Longleftrightarrow & k[t] \arrow[u, hook] \end{tikzcd}$$

<u>Some fibres</u>: $t = a \in k \rightsquigarrow y \in Y$

$y \leftrightarrow (t-a) \subset k[t]$

$$X_y := \operatorname{Spec} k[x,y,t]/(xy-t) \times_Y \operatorname{Spec} k(y)$$

$$= \operatorname{Spec}\left(k[x,y,t]/(xy-t) \otimes_{k[t]} k(y) \right)$$

$$k(y) = \mathcal{O}_{Y,y}/m_y = k[t]_{(t-a)} / (t-a)_{(t-a)} \cong k.$$

$$0 \to (t-a) \to k[t] \to k[t]/(t-a) = k \to 0$$

$k = k(y)$

localize!

$$0 \to (t-a)_{(t-a)} \to k[t]_{(t-a)} \to k \to 0$$

$k = k(y)$.