(b) surjectivity of the map $\varinjlim_{f \notin p} R[f^{-1}] \hookrightarrow R_p$

given $\frac{a}{f^n} \in R_p$ $\quad \frac{a}{f^n}$ is the image of $\frac{a}{f^n} \in R[f^{-1}]$. $\square$

More generally, for any R-module M, we can define the sheaf $\mathcal{M}$ (or $\tilde{M}$) of $\mathcal{O}$-modules on Spec R by setting

$$\mathcal{M}(U_f) := M[f^{-1}] := \text{localization of } M \text{ at } f$$

$$:= M \otimes_R R[f^{-1}]$$

$$\text{or} \ = M \times S / \sim$$

For U arbitrary open set, $\mathcal{M}(U) := \varprojlim_{U_f \subset U} M[f^{-1}]$

As in the case of $\mathcal{O}$, the stalks of $\mathcal{M}$ are

$$\mathcal{M}_\mathfrak{p} \cong \varinjlim_{f \notin \mathfrak{p}} \mathcal{M}(U_f) \cong M_\mathfrak{p} := \text{localization of } M \text{ at } R \setminus \mathfrak{p}$$

$$:= M \times (R \setminus \mathfrak{p}) / \sim$$

$$\text{or} \quad \cong M \otimes_R R_\mathfrak{p}$$

Morphisms of locally ringed spaces:

<u>Recall:</u>

(1) Given two local rings $(A, \mathfrak{m})$, $(B, \mathfrak{n})$, a hom. of rings $\varphi : A \to B$ is called local if $\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}$.

(note: we always have $\varphi^{-1}(\mathfrak{n}) \subset \mathfrak{m}$)

(2) Given a morphism of ringed spaces $(X, \mathcal{O}_X) \xrightarrow{\varphi} (Y, \mathcal{O}_Y)$

$\varphi : X \longrightarrow Y$ cont. map

$\varphi^\# : \mathcal{O}_Y \longrightarrow \varphi_* \mathcal{O}_X$

we have natural maps on the stalks:

$p \in X \qquad \varphi(p) \in Y$

$$\mathcal{O}_{Y,\varphi(p)} \longrightarrow (\varphi_* \mathcal{O}_X)_{\varphi(p)} := \varinjlim_{V \ni \varphi(p)} (\varphi_* \mathcal{O}_X)(V)$$

$$= \varinjlim_{V \ni \varphi(p)} \mathcal{O}_X(\varphi^{-1}(V)) = \varinjlim_{\varphi^{-1}(V) \ni p} \mathcal{O}_X(\varphi^{-1}(V))$$

composition

$$\longrightarrow \varinjlim_{U \ni p} \mathcal{O}_X(U)$$

We say that $\varphi : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a morphism of locally ringed spaces if the above composition is a local hom. of local rings for all $p \in X$.

<u>Def</u>: An affine scheme is a locally ringed space which is isomorphic to the spectrum of a ring.

Given a locally ringed space $(X, \mathcal{O}_X)$, any open set $U \subset X$ has an induced structure of locally ringed space with the topology induced from X and $\forall\ V \subset U$, we define $\mathcal{O}_U(V) := \mathcal{O}_X(V)$

A scheme is a locally ringed space (or just a ringed space) which has a covering by open sets that are affine schemes.

Understanding affine schemes a little better:

Note: The points of $\operatorname{Spec} R$ are not all closed!

In fact: for $\mathfrak{p} \in \operatorname{Spec} R$, the closure $\overline{\{\mathfrak{p}\}} \subset \operatorname{Spec} R$

$$\mathfrak{p} \in \overline{\{\mathfrak{p}\}} = \text{smallest closed set containing } \mathfrak{p}$$

$$= \text{smallest } V(I) \quad \text{"} \quad \text{"}$$

$$= \quad \text{"} \quad V(I) \text{ s.t. } \mathfrak{p} \supset I$$

$$= V(\text{largest } I \text{ s.t. } \mathfrak{p} \supset I)$$

$$= V(\mathfrak{p}) = \{\mathfrak{q} \mid \mathfrak{q} \supset \mathfrak{p}\}$$

$\mathfrak{p}$ is a closed point when $\{\mathfrak{p}\} = \overline{\{\mathfrak{p}\}}$, i.e.,

$\mathfrak{p}$ is the only prime ideal containing $\mathfrak{p}$

$\Leftrightarrow$ $\mathfrak{p}$ is maximal.

So the closed points of $\operatorname{Spec} R$ are the maximal ideals.

For example, for $R = k[x_1, \dots, x_n]$, $\operatorname{Spec} R = \mathbb{A}^n_k$.

We saw that, by Nullstellensatz, when k is algebraically closed, all the maximal ideals are of the form

$$(x_1 - a_1, x_2 - a_2, \dots, x_n - a_n)$$

for some $(a_1, \dots, a_n) \in k^n$.

So, when k is alg. closed, k^n can be identified with the set of closed points of $\mathbb{A}^n_k$.

<u>Back to general schemes and sheaves</u>

<u>Lemma</u> (see Hartshorne) A morphism of sheaves is an

isomorphism iff it is an isomorphism on all the stalks.

A morphism of sheaves is injective if it is injective on sections on all open sets. We will see in homework that this holds iff the morphism is injective on all the stalks.

Given a morphism $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ of sheaves on X, we define the image sheaf $\varphi(\mathcal{F})$ to be the subsheaf of $\mathcal{G}$ which the sheaf associated to the image presheaf $U \mapsto (\operatorname{im} \varphi(U))$.

We say that φ is surjective if the image sheaf of φ is $\mathcal{G}$.

You will see in homework that the datum of a section of the image sheaf $(\mathrm{im}\,\varphi)$ over U is equivalent to the data of an open covering $U = \bigcup_{i \in I} V_i$ and, for each i, a section $t_i = \varphi(s_i) \in \mathrm{im}(\varphi(V_i))$ $(s_i \in \mathcal{F}(V_i))$

(which agree on overlaps)

In Hartshorne: Given two rings A, B. The datum of a morphism $\mathrm{Spec}\,A \to \mathrm{Spec}\,B$ is equivalent to the datum of a hom. of rings $B \to A$.

We will prove something stronger.

First note: Given a morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ we always have a hom. of rings $\mathcal{O}_Y(Y) \longrightarrow \mathcal{O}_X(X)$

recall $\varphi^{\#} : \mathcal{O}_Y \longrightarrow \varphi_* \mathcal{O}_X$

on global sections

$$\begin{tikzcd}
\varphi^{\#}(Y) : \mathcal{O}_Y(Y) \arrow[r] \arrow[ddr, "\varphi^{\#}(Y)"'] & (\varphi_* \mathcal{O}_X)(Y) \arrow[d, phantom, "\parallel"] \\
& \mathcal{O}_X(\varphi^{-1}(Y)) \arrow[d, phantom, "\parallel"] \\
& \mathcal{O}_X(X)
\end{tikzcd}$$

We have the result:

Lemma: given a locally ringed space $(X, \mathcal{O}_X)$ and a ring A, the natural map

$$\mathrm{Hom}(X, \operatorname{Spec} A) \longrightarrow \mathrm{Hom}(A, \mathcal{O}_X(X))$$

$$\varphi \longmapsto \varphi^{\#}(\operatorname{Spec} A)$$

(global sections map)

is a bijection.

Proof: We prove surjectivity first:

We start with a hom. of rings $\alpha : A \to \mathcal{O}_X(X)$

and we construct a morphism $\varphi : X \to \operatorname{Spec} A$ s.t.

$$\alpha = \varphi^{\#}(\operatorname{Spec} A).$$

We first construct the map of sets $\varphi : X \to \operatorname{Spec} A$.

Pick $x \in X$, define $\varphi(x) \in \operatorname{Spec} A$

We have

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[dr, "ev_x \circ \alpha"'] & \mathcal{O}_X(X) \arrow[d, "ev_x"] & \ni & s \arrow[d, mapsto] \\ & \mathcal{O}_{X,x} & \ni & s(x) \\ & \mathfrak{m}_x \arrow[u, phantom, "\cup"] & & \end{tikzcd}$$

$\mathfrak{m}_x$ maximal ideal

$$\varphi(x) := (ev_x \circ \alpha)^{-1}(\mathfrak{m}_x) \subset A$$

prime in A

Now we prove φ is continuous.
We show that $\varphi^{-1}(U_f)$ is open $\forall f \in A$.

$$U_f = \{ p \mid f \notin p \}$$

$$\begin{aligned}
\varphi^{-1}(U_f) &= \{ x \in X \mid \varphi(x) \in U_f \} \\
&= \{ x \in X \mid (ev_x \circ \alpha)^{-1}(m_x) \not\ni f \} \\
&= \{ x \in X \mid (ev_x \circ \alpha)(f) \notin m_x \} \\
&= \{ x \in X \mid (ev_x \circ \alpha)(f) \text{ is invertible} \} \\
&= \{ x \in X \mid \exists\, h_x \in \mathcal{O}_{X,x} \text{ s.t. } h_x (ev_x \circ \alpha)(f) = 1 \} \\
&= \{ x \in X \mid \exists\, U \subset X \text{ open}, x \in U \text{ and } \exists\, h \in \mathcal{O}_X(U) \text{ s.t. } h(x)\alpha(f)(x) = 1 \} \\
&= \{ x \in X \mid \exists\, V \subset X \text{ open}, x \in V \text{ and } h \in \mathcal{O}_X(V) \text{ s.t. } h \cdot \alpha(f)|_V = 1 \}
\end{aligned}$$