Uniqueness of $\tilde{\varphi}$:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{G} \\ \tilde{\mathcal{F}} \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

Given $U \subset X$ and $f \in \tilde{\mathcal{F}}(U)$

$\exists$ open covering $U = \bigcup_{i \in I} W_i$ and sections $s_i \in \mathcal{F}(W_i)$

s.t. $f|_{W_i} = s_i \quad \forall i$, i.e., $\forall x \in W_i$, $f(x) = s_i(x)$.

Then $\forall i$ we have $\varphi(s_i) = \tilde{\varphi}(f|_{W_i}) = \tilde{\varphi}(f)|_{W_i}$

So $\tilde{\varphi}(f)|_{W_i}$ is determined by $\varphi(s_i)$

Since $\mathcal{G}$ is a sheaf $\tilde{\varphi}(f)$ is determined

by $\{ \tilde{\varphi}(f)|_{W_i} \mid i \in I \} = \{ \varphi(s_i) \mid i \in I \}$

$\Rightarrow$ uniqueness.

Existence of $\tilde{\varphi}$: $\quad \tilde{\varphi} : \tilde{\mathcal{F}} \longrightarrow \mathcal{G}$

open $U \subset X$ $\qquad f \in \tilde{\mathcal{F}}(U)$, we define $\tilde{\varphi}(f) \in \mathcal{G}(U)$.

$\exists$ open covering $U = \bigcup_{i \in I} W_i$ and $s_i \in \mathcal{F}(W_i)$ s.t.

$\forall i \qquad f|_{W_i} = s_i$

So $\quad \tilde{\varphi}(f|_{W_i}) = \varphi(s_i) \Rightarrow \tilde{\varphi}(f)|_{W_i} = \varphi(s_i) \in \mathcal{G}(W_i)$

If we know $\boxed{\varphi(s_i)|_{W_i \cap W_j} = \varphi(s_j)|_{W_i \cap W_j}}$

$\forall i, j$, then because $\mathcal{G}$ is a sheaf, $\exists!$ section $\tilde{\varphi}(f) \in \mathcal{G}(U)$ s.t. $\tilde{\varphi}(f)|_{W_i} = \varphi(s_i)$

$\forall \quad x \in W_i \qquad f(x) = s_i(x)$

$\Rightarrow \forall x \in W_i \cap W_j \qquad f(x) = s_i(x)$ and $f(x) = s_j(x)$.

$\Rightarrow \forall x \in W_i \cap W_j \qquad s_i(x) = s_j(x) \in \mathcal{F}_x$.

by the definition of $\mathcal{F}_x$, this means $\exists \; W_x \subset W_i \cap W_j$

s.t. $\qquad s_i|_{W_x} = s_j|_{W_x}$

$\Rightarrow \qquad \forall \; x \in W_i \cap W_j, \; \exists \underset{x}{W_x} \subset W_i \cap W_j$

s.t. $\qquad \varphi(s_i|_{W_x}) = \varphi(s_j|_{W_x})$

$$\varphi(s_i)|_{W_x} = \varphi(s_j)|_{W_x}$$

These W_x form an open covering of $W_i \cap W_j$ and $\mathcal{G}$ is a sheaf, so we can conclude $\quad \varphi(s_i)|_{W_i \cap W_j} = \varphi(s_j)|_{W_i \cap W_j}$. $\square$

<u>Definition</u> (1) <u>Sheaf of rings</u>:

A sheaf $\mathcal{O}_X$ on a topological space X is a sheaf of rings if it is a sheaf s.t. $\forall\ U \subset X$, $\mathcal{O}_X(U)$ is a commutative ring with 1 and $\forall\ V \subset U$, the restriction map $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is a homomorphism of rings.

(2) <u>Sheaf of modules over a sheaf of rings</u>:

Given a sheaf of rings $\mathcal{O}_X$ on X, a sheaf $\mathcal{M}$ of $\mathcal{O}_X$-modules is a sheaf of abelian groups s.t. $\forall$ open $U \subset X$, $\mathcal{M}(U)$ is an $\mathcal{O}_X(U)$-module and $\forall\ V \subset U$, $\forall\ a \in \mathcal{O}_X(U)$, $m \in \mathcal{M}(U)$:

$$(a \cdot m)|_V = a|_V \cdot m|_V .$$

(3) A <u>ringed space</u> is a topological space with a sheaf of rings: $(X, \mathcal{O}_X)$.

(4) A <u>locally ringed space</u> is a ringed space $(X, \mathcal{O}_X)$ s.t. $\forall\, x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.

(5) <u>Push-forward of a sheaf or presheaf:</u>

Given a continuous map $\varphi : Y \to X$ and a presheaf $\mathcal{G}$ on Y, the push-forward $\varphi_* \mathcal{G}$ is the presheaf on X defined by: $\forall$ open $U \subset X$, $(\varphi_* \mathcal{G})(U) := \mathcal{G}(\varphi^{-1}(U))$

<u>exercise</u>: If $\mathcal{G}$ is a sheaf, then so is $\varphi_* \mathcal{G}$.

(6) <u>Morphism of ringed spaces:</u> Given $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ ringed spaces, a morphism $\varphi : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is

the data of a continuous map $\varphi : Y \to X$

and a morphism of sheaves of rings

$$\varphi^{\#} : \mathcal{O}_X \to \varphi_* \mathcal{O}_Y \quad \text{on } X.$$

<u>Main example:</u> Given a ring R

$(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is a ringed space (in fact locally ringed space)

Recall, for $f \in R$ $\quad U_f := \{ \mathfrak{p} \mid f \notin \mathfrak{p} \}$

basic open set

$$\mathcal{O}(U_f) := R[f^{-1}] := R[x] / (fx - 1)$$

the U_f form a basis of the topology of $\operatorname{Spec} R$ and

$$\forall\, U \subset \operatorname{Spec} R \quad \mathcal{O}(U) = \varprojlim_{U_f \subset U} \mathcal{O}(U_f).$$