Note: the constant presheaf is most often not a sheaf.

Examples of sheaves:

(1) X, S two topological spaces.

$\mathcal{F} :=$ sheaf of continuous functions from X to S

$\mathcal{F}(U) := \{$ continuous functions from U to $S \}$

If S is given the discrete topology, then $\mathcal{F}$ is the constant "sheaf" on X with values in S.

(2) Sheaves of C^∞ functions or differential forms on C^∞ manifolds.

(3) sheaves of holomorphic functions on complex analytic spaces, or real analytic functions on real analytic spaces.

(4) $\pi : E \twoheadrightarrow X$ continuous (surjective)

sheaf of continuous sections of π: $U \subset X$

$$\mathcal{F}(U) := \{ s : U \to E \text{ cont.} \mid \pi \circ s = \mathrm{Id}_U \}$$

For k alg. closed, by Nullstellensatz, every maximal ideal in $k[y_1, \dots, y_n]$ is of the form

$$(y_1 - b_1, \dots, y_n - b_n) \text{ for some } (b_1, \dots, b_n) \in k^n.$$

In other words we have a bijection

between $k^n = \{(b_1, \dots, b_n) \mid b_i \in k\}$ and the set of maximal ideals of $A = k[y_1, \dots, y_n]$.

In general, we can always embed

$$k^n \subset \{\text{maximal ideals}\}$$

So, as a first step, we replace affine space with the set of maximal ideals of $k[y_1, \dots, y_n]$

We want to express geometric properties or operations on varieties in terms of their functions, i.e; their coordinate rings.

Classically: $X \xrightarrow{\varphi} Y$ cont. map of top. spaces.

given a cont. function $f: Y \to S$, we obtain a cont. function $X \to S$ by composition:

$$\varphi^*(f) := f \circ \varphi : \begin{tikzcd} X \arrow[r] \arrow[rr, bend right] & Y \arrow[r] & S \end{tikzcd}$$

this is the pull-back of functions.

For affine spaces $\mathbb{A}^m$, $\mathbb{A}^n$, the rings of functions are $k[x_1, \dots, x_m]$ and $k[y_1, \dots, y_n]$.

given a polynomial function $P \in k[y_1, \dots, y_n]$

$P : \mathbb{A}^n \to k$ function.

The pull-back of P is the composition:

$$\mathbb{A}^m \xrightarrow{\varphi} \mathbb{A}^n \xrightarrow{P} k \qquad \underline{\varphi \text{ cont.}}$$

$$\varphi^* P = P \circ \varphi \in k[x_1, \dots, x_m]$$

So $\varphi^* : k[y_1, \dots, y_n] \longrightarrow k[x_1, \dots, x_m]$

$\boxed{\text{ring hom.}}$

condition on φ: φ^* has to send pol. to pol.

$$\varphi : k^m \longrightarrow k^n \qquad \text{(if } k \text{ is alg. closed)}$$

$$a = (a_1, \dots, a_m) \longmapsto (b_1, \dots, b_n) = b$$

If $P(b_1, \dots, b_n) = 0$, then $P \circ \varphi(a_1, \dots, a_m) = 0$

$$\overset{\shortparallel}{(\varphi^* P)}(a_1, \dots, a_m) = 0$$

So $\varphi^*\big(\underbrace{I(b_1, \dots, b_n)}_{(y_1 - b_1, \dots, y_n - b_n)}\big) \subseteq I(a_1, \dots, a_m) = (x_1 - a_1, \dots, x_m - a_m)$

Given a map of algebras $k[y_1, \dots, y_n] \xrightarrow{f} k[x_1, \dots, x_m]$

we can try to define a map $\varphi : \mathbb{A}^m \longrightarrow \mathbb{A}^n$ s.t. $\varphi^* = f$.

Given a maximal ideal m in $k[x_1, \dots, x_m]$, we can try to define $\varphi(m) := f^{-1}(m) \subset k[y_1, \dots, y_n]$.

However: the inverse image of a maximal ideal is not always maximal! It is, however, prime.

The inverse image of any prime ideal is prime!

So we define: $\mathbb{A}^n_k$ affine n-space over a field k, as a set to be $\{ \mathfrak{p} \mid \mathfrak{p} \subset k[y_1, \dots, y_n] \text{ prime} \}$.

$\supset k^n$

Def: R ring (comm. with 1). The spectrum of R as a set is

$$\operatorname{Spec} R := \{ \mathfrak{p} \mid \mathfrak{p} \subset R \text{ prime ideal} \}.$$

Def: The affine space $\mathbb{A}^n_R$ as a set is

$$\mathbb{A}^n_R := \operatorname{Spec} R[x_1, \dots, x_n]$$

Next, we define a topology on $\operatorname{Spec} R$.

For k^n or $\mathbb{P}^n_k$, the closed sets were the sets of zeros of collections of polynomials (or ideals).

For $P \in k[y_1, \dots, y_n]$, $P(b_1, \dots, b_n) = 0$

$$\iff P \in (y_1 - b_1, \dots, y_n - b_n)$$

So, a polynomial vanishes at a maximal or prime ideal $\Leftrightarrow$ it belongs to that ideal

For a general ring, $f \in R$ vanishes at $p \in \operatorname{Spec} R$ iff $f \in p$.

For an ideal $I \subset R$, to say that the elements of I vanish at p means $I \subset p$.

<u>Def</u>: R any ring. The Zariski topology on $\operatorname{Spec} R$ is the topology whose closed sets are

$$V(I) := \{ p \mid I \subset p \} \subset \operatorname{Spec} R$$

for all ideals $I \subset R$.

The functions on Spec R are given by a sheaf of rings.

Recall that for $\mathbb{A}^n_k$, the ring of functions is

$k[y_1, \dots, y_n]$.

So, for Spec R, the ring of functions is R.

Given $f \in k[y_1, \dots, y_n]$, $V(f)$ = zeros of f

Def: $U_f := \mathbb{A}^n_k \setminus V(f)$ is a basic open set.

On U_f, we can divide by f, so the ring of U_f is $k[y_1, \dots, y_n][\frac{1}{f}] := k[y_1, \dots, y_n, X] / (fX - 1)$

On Spec R, we also have "basic" open sets:

$U_f :=$ Spec $R \setminus V(f)$ for $f \in R$

$U_f := \{ \mathfrak{p} \mid \mathfrak{p} \subset R,\ f \notin \mathfrak{p} \}$, $V(f) = \{ \mathfrak{p} \mid f \in \mathfrak{p} \}$

The ring of U_f is, by definition:

$$\mathcal{O}(U_f) := R\left[\frac{1}{f}\right] := R[X] \big/ (fX - 1) .$$

<u>Lemma:</u> Any open set in Spec$R$ is a union of basic open sets. In other words, the basic open sets form a basis of the topology of Spec$R$.

<u>Proof:</u> $U \subset \operatorname{Spec} R$ open. Then $U = \operatorname{Spec} R \setminus V(I)$ for some $I \subset R$. Choose a set of generators $\{f_j\}_{j \in J}$ for I, then $V(I) = \bigcap_j V(f_j)$ and $U = \bigcup_j U_{f_j}$.

$\square$

Lemma: Spec R is quasi-compact.

Proof: Let $\operatorname{Spec} R = \bigcup_{i \in I} V_i$ be an open covering.

By the previous lemma, $\forall i$, we have

$V_i = \bigcup_{j \in J_i} U_{f_{ij}}$ is a union of basic open sets.

So $\operatorname{Spec} R = \bigcup_{i,j} U_{f_{ij}}$

$$\iff \bigcap_{i,j} V(f_{ij}) = \phi$$

$$\bigcap_{i,j} V(f_{ij}) = V(\{f_{ij} \mid i,j\}) = V(\langle f_{ij} \mid i,j \rangle)$$

$V(\langle f_{ij} \mid i,j \rangle) = \phi \iff \nexists\, p$ prime containing $\langle f_{ij} \mid i,j \rangle$

$\iff \langle f_{ij} \mid i,j \rangle = R$

$\iff 1 \in \langle f_{ij} \mid i,j \rangle$

$\iff \exists \quad i_1, \dots, i_n,\ j_1, \dots, j_n, \quad a_1, \dots, a_n \in R$

s.t. $\quad 1 = \sum_{l=1}^{n} a_l f_{i_l j_l}$

$\Rightarrow \quad 1 \in \langle f_{i_1 j_1}, \dots, f_{i_n j_n} \rangle$

$\Rightarrow \quad \bigcap_{l=1}^{n} V(f_{i_l j_l}) = \phi \Rightarrow \quad \operatorname{Spec} R = \bigcup_{l=1}^{n} U_{f_{i_l j_l}}$

$\Rightarrow \quad \operatorname{Spec} R = \bigcup_{l=1}^{n} V_{i_l}$. $\square$

For more details, Eisenbud–Harris: The geometry of schemes.

For an arbitrary open set $U \subset \operatorname{Spec} R$, we write

$$U = \bigcup_{f \in S} U_f \qquad \text{for a suitable } S \subset R.$$

and $$\mathcal{O}(U) = \varprojlim_{f \in S} \mathcal{O}(U_f) \qquad \text{inverse limit}$$

We need to define inverse limits and direct limits.

<u>Def:</u> A directed set is a partially ordered set $(I, \leq)$ s.t. $\forall\, i, j \in I$, $\exists\, k \in I$ s.t. $k \geq i, j$

Given a directed set I and a collection of sets $\{S_i \mid i \in I\}$, we define:

<u>Def</u>: Given a collection of maps $\varphi_{ij} : S_i \longrightarrow S_j$ $\forall\ i \leq j$ s.t. $\forall\ i \leq j \leq k$ $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$,

the direct limit $\varinjlim_{i \in I} S_i$ is

$$\varinjlim_{i \in I} S_i := \coprod_{i \in I} S_i \Big/ \sim$$

where $\underset{\in S_i}{\alpha_i} \sim \underset{\in S_j}{\beta_j}$ if $\exists\ k \geq i, j$ s.t. $\varphi_{ik}(\alpha_i) = \varphi_{jk}(\beta_j)$.

Def: Given a collection of maps $\varphi_{ij} : S_j \to S_i$ $\forall\, i \leq j$ s.t. $\forall\, i \leq j \leq k$ $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$

we define the inverse limit $\varprojlim_{i \in I} S_i$ to be

$$\varprojlim_{i \in I} S_i := \left\{ (\alpha_i)_{i \in I} \;\middle|\; \alpha_i \in S_i \;\; \forall\, i \leq j \;\; \varphi_{ij}(\alpha_j) = \alpha_i \right\}$$

$$\subset \prod_{i \in I} S_i$$

(strictly speaking, we do not need a directed set here, only a partially ordered set)

As we saw, for $U \subset \operatorname{Spec} R$, we define:

Def: $$\mathcal{O}(U) := \varprojlim_{U_f \subset U} R[f^{-1}]$$

the partial order on the set $\{U_f \mid U_f \subset U\}$ is inclusion.

Note: $f, g \in R$ what does it mean for U_f to be contained in U_g?

$$U_f \subset U_g \iff V(f) \supset V(g)$$

$$\iff \{p \mid p \ni f\} \supset \{p \mid p \ni g\}$$

$$\iff \forall\ p \in \operatorname{Spec} R \quad p \ni g \Rightarrow p \ni f$$

$$\iff \bigcap_{p \ni f} p = \sqrt{\langle f \rangle} \subset \bigcap_{p \ni g} p = \sqrt{\langle g \rangle}$$

$\Longleftrightarrow \quad f \in \sqrt{\langle g \rangle}$

$\Longleftrightarrow \quad \exists\, n \quad \text{s.t.} \quad g \mid f^n$

Exercise: Verify that the definition above for $\mathcal{O}(U)$ indeed defines a sheaf on $\operatorname{Spec} R$. (also see "Geometry of schemes")

We are going to see this is a sheaf in a different way, using the "espace étalé".

Def: The stalk of a presheaf $\mathcal{F}$ on X at a point $x \in X$ (for any topological space X) is

$$\mathcal{F}_x := \varinjlim_{x \in U \subset X} \mathcal{F}(U)$$