For $j \in \{0, \dots, n-1\}$, we show $D(\mathfrak{p}_j) \not\supseteq D(\mathfrak{p}_{j+1})$

we know $\mathfrak{p}_j \not\supseteq \mathfrak{p}_{j+1}$, choose $a \in \mathfrak{p}_j \setminus \mathfrak{p}_{j+1}$
a homogeneous.

if $D(a) \in D(\mathfrak{p}_{j+1})$, then $\exists\, b \in \mathfrak{p}_{j+1}$ s.t. $D(a) = D(b)$

$\Rightarrow \exists r > 0$ s.t. $a = x_0^r b$ or $b = x_0^r a \in \mathfrak{p}_{j+1}$

$$\begin{array}{ccc} a = x_0^r b & & b = x_0^r a \in \mathfrak{p}_{j+1} \\ \Downarrow & & \Downarrow \quad a \notin \mathfrak{p}_{j+1} \\ a \in \mathfrak{p}_{j+1} & & x_0 \in \mathfrak{p}_{j+1} \subset I(Y) \\ \text{not allowed} & & \Downarrow \\ & & x_0 \in I(Y) \Rightarrow Y \subset Z(x_0) \\ & & \Rightarrow Y \cap V_0 = \emptyset \\ & & \text{not allowed.} \end{array}$$

So all the inclusions are strict, and
height $I(Y) \le$ height $I(Y \cap V_0)$ $\square$

Lemma: A subset $Y \neq \emptyset$ of a topological space $X$ is irreducible iff $\forall\ Y_1, Y_2$ closed in $X$,

$$Y \subset Y_1 \cup Y_2 \implies Y \subset Y_1 \text{ or } Y \subset Y_2$$

($Y$ is given the topology induced from $X$)

Proof: exercise.

Proposition: In a noetherian topological space $X$, any closed subset $Y \neq \emptyset$ is a finite union $Y_1 \cup \cdots \cup Y_r$ of irreducible closed subsets. If $\forall\ i \neq j$, $Y_i \not\subset Y_j$, then the decomposition is unique up to reindexing of the $Y_i$.

Definition: The $Y_i$ as above are called the irreducible components of $Y$.

<u>Proof:</u> <u>Existence</u> Part

$\mathfrak{S} := \{$ non-empty closed subsets that are NOT finite unions of irreducible closed subsets $\}$

If $\mathfrak{S} \neq \phi$, then, because $X$ is noetherian, every descending chain of elements of $\mathfrak{S}$ has a minimal element.
So, by Zorn's lemma, $\mathfrak{S}$ has a minimal element, say $Y$.

$Y \in \mathfrak{S} \Longrightarrow Y$ is not irreducible

$\Rightarrow \exists\, Y_1, Y_2$ closed, not empty in $Y$ s.t.

$Y = Y_1 \cup Y_2$, $Y_1 \subsetneq Y$ and $Y_2 \subsetneq Y$

Since $Y$ is minimal in $\mathfrak{S}$, $Y_1$ and $Y_2 \notin \mathfrak{S}$.

$\Rightarrow Y_i =$ finite union of irreducible closed subsets

$\Rightarrow$ $Y =$ finite union of irred. closed subsets

$\Rightarrow$ $Y \notin \mathfrak{S}$ contradiction.

Uniqueness up to reindexing:

Suppose $Y = Y_1 \cup \cdots \cup Y_r = Y_1' \cup \cdots \cup Y_s'$

$Y_i \not\subset Y_j$ $\quad$ $Y_i' \not\subset Y_j'$

$i \neq j$ $\quad$ $i \neq j$

$Y_r$ is irreducible and contained in $Y_1' \cup \cdots \cup Y_s'$,

by the lemma, $\exists\, j$ s.t. $Y_r \subset Y_j'$

similarly, $\exists\, i$ s.t. $Y_j' \subset Y_i$ $\Rightarrow Y_r \subset Y_i$

$\Rightarrow$ $i = r$ and $Y_j' \subset Y_r \Rightarrow Y_j' = Y_r$

repeat to obtain that $\forall i, \exists j$ s.t. $Y_i = Y'_j$ and $r = s$. $\square$

<u>Remark</u>: Did we need Zorn's lemma?

If $\mathfrak{S}$ does not have a minimal element, then we can produce an infinite descending chain of closed subsets of $X$. This is still the axiom of choice.

__Example:__ $Y \subset \mathbb{A}^n$ a hypersurface

by def., $\exists\, f \in A = k[y_1, \dots, y_n]$ s.t. $Y = Z(f) = Z(\langle f \rangle)$

Then $I(Y) = \sqrt{\langle f \rangle}$ by Nullstellensatz

$\langle f \rangle$ = ideal generated by $f$.

We can write $f = g_1^{r_1} g_2^{r_2} \cdots g_m^{r_m}$ where $g_i$ are irreducible polynomials and $r_i > 0$ and $g_i$ do not divide each other.

Then $\sqrt{\langle f \rangle} = \langle g_1 \cdots g_m \rangle$

Recall $\sqrt{I} = \bigcap\limits_{\mathfrak{p} \supset I} \mathfrak{p}$ for any ideal

The set of minimal prime ideals containing $f$ is $\{\langle g_1 \rangle, \dots, \langle g_m \rangle\}$

$$\sqrt{\langle f \rangle} = \bigcap_{i=1}^{m} \langle g_i \rangle = \langle g_1 \cdots g_m \rangle$$

So $\quad Z(f) = \bigcup_{i=1}^{m} Z(g_i)$

irreducible closed in $\mathbb{A}^n$

$Z(g_i) \not\subset Z(g_j)$ for $i \neq j$

because $\langle g_j \rangle \not\subset \langle g_i \rangle$

so the $Z(g_j)$ are the irreducible components of

$Y = Z(f)$.

$Y$ is irreducible iff $m=1$, meaning $f$ is a prime power.

$f = g_1^{r_1}$, $\quad I(Y) = \langle g_1 \rangle$

$\dim Y = \dim A(Y) = \dim A/\langle g_1 \rangle$

$= \dim A - \text{height}\langle g_1 \rangle$

<u>Prop. 1.11A:</u> For any $f \in A$ noetherian ring, if $f$ is not a zero divisor, then the height of any minimal prime containing $f$ is 1.

$\Rightarrow$ height$\langle g, \rangle = 1$

and $\dim Y = \dim A - 1 = n-1$.

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<u>A few words about cones</u> (good sources of examples): also further link projective to affine

<u>The affine cone over a projective alg. set:</u>

Let $\theta : \mathbb{A}^{n+1} \setminus \{0\} \twoheadrightarrow \mathbb{P}^n$ be the quotient map.

For any projective alg. set $Y \subset \mathbb{P}^n$, define affine cone over $Y$ to be $C(Y) := \theta^{-1}(Y) \cup \{0\} \subset \mathbb{A}^{n+1}$

By definition the ideal of $C(Y)$ as an affine alg. set is equal to the homogeneous ideal of $Y$ as a projective alg set:

$$\begin{array}{ccc} I(C(Y)) & = & I(Y) \\ \cap & & \cap \\ k[x_0,\dots,x_n] & = & k[x_0,\dots,x_n] = S \\ \text{coordinate ring of } \mathbb{A}^{n+1} & & \text{coordinate ring of } \mathbb{P}^n \end{array}$$

<u>The projective cone over a projective alg. set:</u>

$Y$ and $C(Y)$ as above: Choose an embedding

$$\mathbb{A}^{n+1} \hookrightarrow \mathbb{P}^{n+1} \quad \text{e.g. as } U_{n+1} = \mathbb{P}^{n+1} \setminus Z(x_{n+1})$$

$$(b_0,\dots,b_n) \longmapsto (b_0,\dots,b_n, 1)$$

The projective cone $\overline{C(Y)}$ of $Y$ is the closure of $C(Y)$ in $\mathbb{P}^{n+1}$ via an embedding as above.

$$\mathbb{A}^{n+1} \xrightarrow{\cong} U_{n+1} \subset \mathbb{P}^{n+1}$$

The ideal of $\overline{C(Y)}$ in $\mathbb{P}^{n+1}$ is the homogenization of $I(C(Y))$ in $k[x_0, \ldots, x_{n+1}]$ with respect to $x_{n+1}$.

This is just $I(C(Y))$ because $I(C(Y))$ is already homogeneous.

## Sheaves

**Presheaves**: **Def**: Let $X$ be a topological space. A presheaf $\mathcal{F}$ (of sets) on $X$ is the data,

(1) for each open set $U \subset X$, of a set $\mathcal{F}(U)$,

(2) for every inclusion of open sets $V \subset U \subset X$, of

a restriction map $\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$,

such that:

(1) $\mathcal{F}(\emptyset) = \{\emptyset\}$ the set with one element,

(2) $\rho_{UU} = Id_U$ the identity map of $\mathcal{F}(U)$,

(3) for all inclusions of open sets $W \subset V \subset U$, we have $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\rho_{UV}"] \arrow[rr, bend right, "\rho_{UW}"'] & \mathcal{F}(V) \arrow[r, "\rho_{VW}"] & \mathcal{F}(W) \end{tikzcd}$$

<u>Note</u>: A presheaf is a contravariant functor from the category of open sets of $X$ to the category of sets.

The target category can also be , abelian groups, rings, modules over a fixed ring, etc. (maps of sets would have to be changed to morphisms in each target category), $\{\phi\}$ would have to be replaced by a final object.

<u>Example</u>: Constant presheaves: $\mathcal{F}(U) = S$ for a fixed set $S$, $\forall\ U \neq \phi$.

<u>Notation</u>: From now on, we denote $\rho_{UV}: \mathcal{F}(U) \to \mathcal{F}(V)$ by $|_V := \rho_{UV}$.

<u>Definition</u>: A presheaf $\mathcal{F}$ on $X$ is a sheaf if it satisfies the following.

(1) for all open sets $U$ and all open coverings

$$U = \bigcup_{i \in I} V_i$$ , and, for all $s, t \in \mathcal{F}(U)$ "sections"

of $\mathcal{F}$ over $U$, if $s|_{V_i} = t|_{V_i}$ $\forall i \in I$, then $s = t$.

(2) for all $U$, all $U = \bigcup_{i \in I} V_i$, and all

collections $\{ s_i \in \mathcal{F}(V_i) \mid i \in I \}$ s.t. $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$

$\forall i, j$,

$\exists \ s \in \mathcal{F}(U)$ s.t. $s|_{V_i} = s_i \ \forall i$.