$$\dim(R) = \sup\{ n \mid \exists\ \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subsetneq R \}$$
$$\mathfrak{p}_i \text{ prime}$$

<u>Prop (1.8A)</u> : R an integral domain which is also a finitely generated k-algebra, then

$$\dim(R/\mathfrak{p}) + \operatorname{height}(\mathfrak{p}) = \dim R$$

$\forall$ prime ideals $\mathfrak{p} \subset R$.

Also $\quad \dim R = \operatorname{trdeg}_k \operatorname{Frac}(R)$, also denote $K(R) := \operatorname{Frac}(R)$

<u>recall</u>: for any extension of fields $K \subset L$,

$$\operatorname{trdeg}_K L = \sup\{ n \mid \exists\ x_1, \dots, x_n \in L \text{ algebraically independent over } K \}$$

e.g. $A = k[y_1, \dots, y_n]$, then $K(A) = k(y_1, \dots, y_n)$

$$\operatorname{trdeg}_k K(A) = n$$

or $A(Y) = k[x,y]/(y-x^2)$, if $Y = \text{parabola} = Z(y-x^2)$

$\cong k[x]$ $\qquad K(A(Y)) = \operatorname{Frac}\left(k[x,y]/(y-x^2)\right)$

$$\cong k(x)$$

$$\operatorname{trdeg}_k K(A(Y)) = 1.$$

<u>Proof of the theorem:</u> First note that, by the lemma: Y irreducible $\iff I(Y)$ prime

$\iff A(Y) = A/I(Y)$ is an integral domain

Proof of $\dim Y = \dim A(Y)$ for Y affine irreducible:

The data of a chain of irreducible closed subsets of Y: $Y \supseteq Y_0 \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_n \neq \phi$

is equivalent (by Nullstellensatz) to the data of the chain of prime ideals

$$I(Y) \subseteq I(Y_0) \subsetneq I(Y_1) \subsetneq \cdots \subsetneq I(Y_n) \subsetneq A = k[y_1, \dots, y_n]$$

$$\Longleftrightarrow \quad (0) \subseteq \frac{I(Y_0)}{I(Y)} \subsetneq \frac{I(Y_1)}{I(Y)} \subsetneq \cdots \subsetneq \frac{I(Y_n)}{I(Y)} \subsetneq \frac{A}{I(Y)} = A(Y)$$

$$\Rightarrow \quad \dim Y = \dim A(Y)$$

Proof of $\dim Y = \dim S(Y) - 1$ for Y irreducible projective

The data of a chain of irreducible closed subsets of Y:

$$Y \supseteq Y_0 \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_n \supsetneq \phi$$

is equivalent to the data of their homogeneous prime ideals (by homogeneous Nullstellensatz):

$$I(Y) \subseteq I(Y_0) \subsetneq I(Y_1) \subsetneq \cdots \subsetneq I(Y_n) \subsetneq S_+ \subsetneq S$$

$$\Longleftrightarrow \quad (0) \subseteq \frac{I(Y_0)}{I(Y)} \subsetneq \frac{I(Y_1)}{I(Y)} \subsetneq \cdots \subsetneq \frac{I(Y_n)}{I(Y)} \subsetneq S(Y)_+ \subsetneq S(Y)$$

$$\Rightarrow \quad \dim Y = \operatorname{height} S(Y)_+ - 1$$

$$= \dim S(Y) - \dim S(Y)/S(Y)_+ - 1$$

$$\left. S(Y) \middle/ S(Y)_+ \right. = \left. {}^{S}/_{I(Y)} \middle/ {}^{S_+}/_{I(Y)} \right. \cong S/S_+ \cong k$$

$$\Rightarrow \quad \dim Y = \dim S(Y) - \text{Krull dim } k - 1 = \dim S(Y) - 1$$

Krull dim$(k) = 0$ because k has exactly one prime ideal: $(0) \subset k$.

$\square$

$$A(k) = A(\mathbb{A}^1) = k[y_1]$$

$$1 = \dim \mathbb{A}^1 = \text{Krull dim } k[y_1]$$

Recall that $\mathbb{P}^n = \bigcup_{i=0}^{n} U_i$ $\quad U_i = \operatorname{Im} \varphi_i$

$\varphi_i : \mathbb{A}^n \hookrightarrow \mathbb{P}^n$

$(t_1, \dots, t_n) \longmapsto (t_1, \dots, \underset{\text{i-th coord.}}{1}, \dots, t_n)$

$U_i = \mathbb{P}^n \setminus Z(x_i)$

<u>Lemma:</u> For $Y \subset \mathbb{P}^n$ projective, irred., $\forall i \in \{0, \dots, n\}$

if $Y \cap U_i \neq \emptyset$, then $\dim Y = \dim (Y \cap U_i)$

<u>Proof</u>: WLOG assume $i=0$.

By the theorem, $\dim Y = \dim S(Y) - 1$

$= \dim S - \text{height}\, I(Y) - 1$

$= \text{tdeg}_k\, k(x_0, \dots, x_n) - \text{height}\, I(Y) - 1$

$= n + 1 - \text{height}\, I(Y) - 1$

$= n - \text{height}\, I(Y)$

Similarly $Y \cap U_0 \subset U_0 \underset{\varphi_0}{\cong} \mathbb{A}^n$ homeomorphism

By the theorem, $\dim(Y \cap U_0) = \dim A(Y \cap U_0)$

$$= \dim A - \operatorname{height} I(Y \cap U_0)$$

$$= \operatorname{trdeg}_k k(y_1, \dots, y_n) - \operatorname{height} I(Y \cap U_0)$$

$$= n - \operatorname{height} I(Y \cap U_0)$$

So $\dim Y = \dim(Y \cap U_0) \iff \operatorname{height} I(Y) = \operatorname{height} I(Y \cap U_0)$

We have $I(Y) \subset S = k[x_0, \dots, x_n]$

$$I(Y \cap U_0) \subset A = k[y_1, \dots, y_n]$$

$$\varphi_0 : \mathbb{A}^n \xrightarrow{\cong} U_0 \subset \mathbb{P}^n$$

$$(b_1, \dots, b_n) \longmapsto (1, b_1, \dots, b_n)$$

$$\left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right) \longleftarrow (a_0, \dots, a_n) \quad a_0 \neq 0$$

We have $H : A \longrightarrow S$ homogenization

$$P \longmapsto H(P) := x_0^d P\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$$

$$P = P(y_1, \dots, y_n) \qquad d := \deg P$$

and dehomogenization $D : S \longrightarrow A$

$$Q \longmapsto D(Q) := Q(1, y_1, \dots, y_n)$$
$$\overset{\shortparallel}{Q(x_0, \dots, x_n)}$$

Check: $D(I(Y)) = I(Y \cap U_0)$, $I(Y) = H(I(Y \cap U_0))$

exercise

true because $Y \cap U_0 \neq \phi$ and Y is irreducible.

where $D(I(Y))$:= ideal generated by the dehomogenizations of all the elements of $I(Y)$

$H(I(Y \cap U_0))$:= ideal generated by the homogenizations of all the elements of $I(Y \cap U_0)$

To show that $I(Y)$ and $I(Y \cap U_0)$ have the same height, we show that we have a bijection between their sets of chains of prime ideals.

Given $I(Y \cap U_0) \supseteq \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_m$

homogenize to get $I(Y) \supseteq H(\mathfrak{p}_0) \supsetneq H(\mathfrak{p}_1) \supsetneq \cdots \supsetneq H(\mathfrak{p}_m)$

the inclusions are strict because $D(H(P)) = P \ \forall P$

$$\Rightarrow D(H(\mathfrak{p}_i)) = \mathfrak{p}_i$$

$$\Rightarrow \text{height } I(Y) \geq \text{height } I(Y \cap U_0)$$

Now start with $I(Y) \supseteq \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_m$

dehomogenize to get $I(Y \cap U_0) \supseteq D(\mathfrak{p}_0) \supseteq D(\mathfrak{p}_1) \supseteq \cdots \supseteq D(\mathfrak{p}_m)$

need to show that the inclusions are strict.

$$H(D(Q)) = x_0^r Q \quad \text{for some } r \leq 0$$

e.g. $x_0^2 x_1 + x_0 x_3^2 \overset{dh}{\rightsquigarrow} y_1 + y_3^2 \overset{h}{\rightsquigarrow} x_0 x_1 + x_3^2$