Algebraic geometry is the study of geometric objects using algebra (in practice, we use a lot more than algebra, we use topology, we use analysis, diff. geom. etc).

Our guiding principle is that geometric objects are determined by their functions and all of their properties are reflected in the properties of their functions. This is a point of view that has revolutionized modern mathematics (it is used in many fields).

Algebro-geometric objects are characterized by the fact that their functions are algebraic, i.e., "rational functions".

Simplest algebro-geometric objects:

affine spaces. $\longleftrightarrow$ functions are polynomials.

ring of functions $=$

$k[x_1, \dots, x_n]$

k a field

In a first approximation, n-dimensional affine space over k is k^n. $x_1, \dots, x_n$ are then the coordinate functions on k^n.

We define the Zariski topology on k^n:

Def: Given a set $T \subset k[x_1, \dots, x_n]$

define $Z(T) := \{(a_1, \dots, a_n) \in k^n \mid P(a_1, \dots, a_n) = 0 \ \forall P \in T\}$

Def and Proposition: The collection of sets $Z(T)$ for $T \subset k[x_1, \dots, x_n]$ is a topology on k^n. This is called the Zariski topology.

Proof: Verify: (1) ϕ is closed: $\phi = Z(T)$ where $T = \{1\}$

(2) k^n is closed: $k^n = Z(T)$ where $T = \{0\}$

(3) arbitrary intersection of closed sets is closed:

$\{T_i,\ i\in I\}$ $\quad T_i \subset k[x_1,\dots,x_n]$

$$\bigcap_{i\in I} Z(T_i) = Z\Big(\bigcup_{i\in I} T_i\Big)$$

(4) finite unions of closed sets are closed:

$P_1 \in k[x_1,\dots,x_n]$ $\quad P_2 \in k[x_1,\dots,x_n]$

$$Z(P_1)\cup Z(P_2) = Z(P_1 P_2)$$

$$Z(T_1)\cup Z(T_2) = Z(T_1 T_2)$$

where $T_1 T_2 := \{P_1 P_2 \mid P_1 \in T_1,\ P_2 \in T_2\} \subset k[x_1,\dots,x_n]$

$$Z(T_1)\cup Z(T_2)\cup\dots\cup Z(T_m) = Z(T_1\cdots T_m)$$

$T_1\cdots T_n := \{P_1\cdots P_m \mid P_i \in T_i\} \subset k[x_1,\dots,x_n]$ $\qquad\square$

Def and Prop.: An affine algebraic set is a closed subset of an affine space. The ideal of an affine algebraic set Y is

$$I(Y) := \{ P \in k[x_1, \dots, x_n] \mid P(a_1, \dots, a_n) = 0 \ \forall (a_1, \dots, a_n) \in Y \}$$

Ex: verify that this is indeed an ideal.

The functions on Y are the restrictions to Y of polynomials.

The values of a polynomial $P \in k[x_1, \dots, x_n]$ on Y depend only on the class of P modulo $I(Y)$, i.e., on

$$\bar{P} \in k[x_1, \dots, x_n] / I(Y).$$

So the functions on Y are the elements of $A(Y) := k[x_1, \dots, x_n] / I(Y)$

<u>Def</u>: $A(Y)$ is called the coordinate ring of Y.

The topology of k^n induces a topology on Y: this is the Zariski topology on Y, its closed sets are the sets of zeros of collections $T \subset A(Y)$.

After affine spaces, the simplest objects are projective spaces.

<u>Def</u>: $\mathbb{P}^n_k := \left(k^{n+1} \setminus \{0\}\right) / k^*$

where $k^* := k \setminus \{0\}$ acts on k^{n+1} via scalar multiplication:

$$\lambda \in k^* \qquad \lambda(a_0, \dots, a_n) = (\lambda a_0, \dots, \lambda a_n)$$

equivalently, $\mathbb{P}^n_k$ is the set of lines through the origin in k^{n+1}.

<u>Functions on $\mathbb{P}^n_k$?</u> first idea: $k[x_0, \dots, x_n]$

polynomials do not give well-defined functions on $\mathbb{P}^n_k$:

e.g. $p(x_0, x_1) = x_0 + x_1^3$ value on the class of $(a_0, a_1) = (\lambda a_0, \lambda a_1) \in \mathbb{P}^1_k$

However, we can still define a Zariski topology:

We can define sets of zeros of "homogeneous" polynomials.

<u>Def</u>: A polynomial is homogeneous if it is a k-linear combination of monomials of the same degree.

e.g. $P(x_0, x_1) = x_0 + x_1^3$ is <u>not</u> homogeneous.

$Q(x_0, x_1) = x_0 x_1^2 - x_0^3 + x_1^3$ is homogeneous.

<u>Prop</u>: P is homogeneous of degree d iff $\forall \lambda \in k^*, a_0, \dots, a_n \in k,$

$P(\lambda a_0, \dots, \lambda a_n) = \lambda^d P(a_0, \dots, a_n)$. (Proof: exercise)

Zeros of homogeneous polynomials are "cones" in k^{n+1}: unions of lines through the origin.

So, if P is homogeneous, then $Z(P) \subset \mathbb{P}^n_k$ is well-defined.

<u>Definition:</u> Given $T \subset k[x_0, \dots, x_n]$ s.t. all $P \in T$ are homogeneous, define

$$Z(T) := \{ (a_0, \dots, a_n) \in \mathbb{P}^n \mid P(a_0, \dots, a_n) = 0 \ \forall P \in T \}$$

<u>Def. and proposition:</u> The Zariski topology on $\mathbb{P}^n$ is the topology whose closed sets are the $Z(T)$ for $T \subset k[x_0, \dots, x_n]$ a set of homogeneous polynomials.

Proof: Verify: (1) $\phi = Z(1)$

$= Z(\{x_0, x_1, \dots, x_n\})$

(2) $\mathbb{P}^n = Z(0)$

(3) $\bigcap_{i \in I} Z(T_i) = Z\left(\bigcup_{i \in I} T_i\right)$

(4) $Z(T_1) \cup \cdots \cup Z(T_m) = Z(T_1 \cdots T_m)$

(Ex: a product of homogeneous polynomials is homogeneous).

Def: A projective algebraic set Y is the set of zeros of a set of homogeneous polynomials in $\mathbb{P}^n_k$.

It will inherit the Zariski topology from $\mathbb{P}^n_k$.

The closed sets of Y are the sets of zeros of homogeneous polynomials in Y.

The zero set of a homogeneous polynomial in Y depends only on the class of the polynomial modulo the polynomials that vanish on Y.

<u>Def</u>: $I(Y) :=$ ideal generated by the homogeneous polynomials vanishing on Y.

$S(Y) := k[x_0, \dots, x_n] / I(Y)$ is the homogeneous coordinate ring of Y.

So the closed subsets of Y are zeros of classes of homogeneous polynomials in $S(Y)$.

$I(Y)$ is a homogeneous ideal:

Def: An ideal $I \subset k[x_0, \dots, x_n]$ is homogeneous if it can be generated by homogeneous polynomials.

$S := k[x_0, \dots, x_n]$ is a graded ring, i.e.,

$$S = \bigoplus_{d \geq 0} S_d \quad \text{where } S_d := \{\text{homogeneous pol. of degree } d\} \cup \{0\}$$

Note: S_d is an abelian subgroup of S, but not a subring.

In fact: $S_d \cdot S_e \subset S_{de}$.

Def: For an ideal $I \subset k[x_0, \dots, x_n] =: S$

we denote $I_d := I \cap S_d \subset I$

$$\Rightarrow \bigoplus_{d \geq 0} I_d \subset I$$

Lemma: I is homogeneous iff $I = \bigoplus_{d \geq 0} I_d$.

Proof: exercise

Link between affine and projective:

$$\mathbb{A}^n_k = k^n \qquad \qquad \mathbb{P}^n_k = k^{n+1} \setminus \{0\} / k^*$$

$\forall\, i \in \{0, \dots, n\}$, we have injective maps

$$\varphi_i : \mathbb{A}^n \hookrightarrow \mathbb{P}^n_k$$

$$(y_1, \dots, y_n) \longmapsto (y_1, \dots, \underset{\text{i-th coordinate}}{1}, \dots, y_n)$$

proof of injectivity: exercise.

We denote the image of φ_i by U_i:

this is the locus where $x_i = 1 \iff x_i \neq 0$ (on $\mathbb{P}^n$)

$$U_i = \{(a_0, \dots, a_n) \in \mathbb{P}^n \mid a_i \neq 0\}$$

because $(a_0, \dots, a_i, \dots, a_n) \sim \left(\frac{a_0}{a_i}, \dots, 1, \dots, \frac{a_n}{a_i}\right)$

if $a_i \neq 0$.

Note: $\mathbb{P}^n = \bigcup_{i=0}^{n} U_i$ $\left(\mathbb{P}^n = k^{n+1} \setminus \{0\} / k^*\right)$

$\forall\ (a_0, \dots, a_n)$, $\exists i$ s.t. $a_i \neq 0$

$\Rightarrow (a_0, \dots, a_n) \in U_i$.

Also if $Y_i := \mathbb{P}^n \setminus U_i$

then $Y_i = \{(a_0, \dots, a_n) \mid a_i = 0\} = Z(x_i)$

is closed $\Rightarrow U_i$ is open.