Some properties of separated and proper morphisms:

(1) Open and closed embeddings are separated.
Closed embeddings are proper.

(2) Compositions of separated, resp. proper, morphisms are separated, resp. proper.

(3) If $f: X \to Y$ and $g: Y \to Z$ are morphisms, then, if $(g \circ f)$ is separated, then f is separated.

(4) If $f: X \to Y$, $g: Y \to Z$ are morphisms, $g \circ f$ is proper and g is separated, then f is proper.

(5) The properties of being separated or proper is local on the base, i.e., $f: X \to Y$ is separated, resp. proper,

iff $\exists$ covering $Y = \bigcup_{i \in I} V_i$

s.t. $\forall i$ $f|_{V_i} : f^{-1}(V_i) \to V_i$ is separated,
resp. proper.

(6) separated and proper morphisms are stable under base change, i.e., $\forall$ $f : X \to Y$, $\forall$ $g : Y' \to Y$

form

$$\begin{tikzcd} X' = X \times_Y Y' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

f separated $\Rightarrow$ f' separated
f proper $\Rightarrow$ f' proper.

(7) Products of separated, resp. proper, morphisms are separated, resp. proper. i.e., given

$f: X \longrightarrow Y (\longrightarrow S)$ $\quad g: X' \longrightarrow Y' (\longrightarrow S)$

then f, g separated $\Rightarrow$ $f \times g : X \times_S X' \longrightarrow Y \times_S Y'$ is separated

f, g proper $\Rightarrow$ $f \times g$ proper

Sheaves of modules:

Throughout, all schemes are noetherian.

(1) recall the definition of a sheaf of modules on a scheme X: it is a sheaf of abelian groups s.t.

$\forall$ open $U \subset X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module.

and the module structures are compatible with the restriction morphisms: $\forall$ $V \subset U \subset X$ open sets

$\forall$ $a \in \mathcal{O}_X(U)$, $s \in \mathcal{F}(U)$: $(as)\big|_V = a\big|_V \cdot s\big|_V$

(2) A morphism of $\mathcal{O}_X$-modules $\varphi: \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves s.t. $\forall$ $U \subset X$ open, $\varphi(U): \mathcal{F}(U) \to \mathcal{G}(U)$ is a hom. of $\mathcal{O}_X(U)$-modules.

(3) The kernel, cokernel and image of a morphism of $\mathcal{O}_X$-modules is again an $\mathcal{O}_X$-module. The quotient of two $\mathcal{O}_X$-modules is an $\mathcal{O}_X$-module (cokernel of an inclusion).

(4) The direct sum, direct product, direct limit, inverse limit of $\mathcal{O}_X$-modules are again $\mathcal{O}_X$-modules.
(in these cases the presheaves are already sheaves, for direct limits, we need the noetherian hyp. Ex. in Section 1)

(5) <u>Tensor products</u>: Given $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, the tensor product $\mathcal{F} \otimes \mathcal{G}$ is the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. It is an $\mathcal{O}_X$-module.

(6) The sheaf $\mathcal{H}om$:

Given two $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, we have the set

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) := \left\{ \begin{array}{c} \text{morphisms of } \mathcal{O}_X\text{-modules} \\ \mathcal{F} \longrightarrow \mathcal{G} \end{array} \right\}$$

The sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$

the presheaf $\quad U \longmapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$.

$\mathcal{H}$ is an $\mathcal{O}_X$-module (and already a sheaf).

(7) An $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if there exist an open cover of X by affine subschemes $U = \mathrm{Spec}\, A$ s.t. $\exists$ an A-module M with $\mathcal{F}|_U \cong \tilde{M}$.

<u>recall:</u> $\tilde{M}$ is the sheaf on $\mathrm{Spec}\, A$ s.t. $\forall$ basic open $\mathrm{Spec}\, A[f^{-1}]$

$\tilde{M}(\operatorname{Spec} A[f^{-1}]) = M[f^{-1}]$.

One can prove that then this holds for any open affine subset of X.

A quasi-coherent sheaf is called coherent if, in addition, M is a finite A-module.

(8) An $\mathcal{O}_X$-module is called free if it is isomorphic, as an $\mathcal{O}_X$-module, to a direct sum of sheaves isomorphic to $\mathcal{O}_X$ (sometimes called trivial).

It is locally free if X has a covering by open sets U s.t. $\mathcal{F}|_U$ is free.

An isomorphism $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus I}$ is called a trivialization of $\mathcal{F}$ on U.

The rank of a locally free sheaf is the number of copies of $\mathcal{O}_X$ in a trivialization, it can be finite or infinite. The rank of a locally free sheaf is constant on each connected component of X.