Remark: \[ \text{Spec } R = T \hookrightarrow \mathbb{Z} \hookrightarrow X \]

integral schemes: have generic points: \( t_1, x_1 \)

\[ \mathcal{O}_{\mathbb{Z}, x_1} = \text{field } \supset \mathcal{O}_{\mathbb{Z}, t'_1} \quad \forall t'_1 \in \mathbb{Z} \]

from the first part \( k(x_1) \subset K \)

verify \( \mathcal{O}_{\mathbb{Z}, x_1} \cup \mathcal{O}_{\mathbb{Z}, x_0} \subset R = \mathcal{O}_{T_1, t_1} \)

verify \( \mathcal{O}_{\mathbb{Z}, x_0} \subset R \) dominates \( \mathcal{O}_{\mathbb{Z}, x_0} \) means \( M_R \cap \mathcal{O}_{\mathbb{Z}, x_0} = M_{\mathbb{Z}, x_0} \).

Proof of lemma continued:

Conversely, suppose we are given \( x_0, x_1 \in X, x_0 \in \mathbb{Z} = \overline{\{ x_1 \}} \)

and the inclusion \( k(x_1) \subset K \) s.t. \( R \) dominates \( \mathcal{O}_{\mathbb{Z}, x_0} \).
The inclusion \( \mathbb{G}_m, x_0 \to R \) gives a morphism
\( T \to \text{Spec } \mathbb{G}_m, x_0 \). Compose this with
\( \text{Spec } \mathbb{G}_m, x_0 \to X : \)
Given any affine neighborhood \( U = \text{Spec } A \subset X \) of \( x_0 \),
then \( U \) also contains \( x_1 \) because \( x_0 \in \{ x_1 \} \)
\( x_i \in X \setminus U \implies \{ x_1 \} \subset X \setminus U \implies x_0 \in X \setminus U \)
\( x_i \in U \iff x_0 \in U \)
We now have the evaluation map \( A = \mathcal{O}(U) \to \mathcal{O}_{X, x_0} \)
\( \Rightarrow \text{Spec } \mathbb{G}_m, x_0 \to \text{Spec } \mathcal{O}_{X, x_0} \to \text{Spec } A = U \)
\( \Rightarrow T \to \text{Spec } \mathbb{G}_m, x_0 \to \text{Spec } A \to X \)
Some terminology: When $x_0 \in \{x_1\}$, we say
- $x_0$ is a specialization of $x_1$
- $x_1$ is a generalization of $x_0$

Proof of the valuative criterion of separatedness:

Assume $f: X \to Y$ is separated.

Suppose we have $\text{Spec} K = U \to X$ and two morphisms $h, h'$ making the diagram commutative. We show $h = h'$. 
\[ h|_U = h'|_U \]

**Claim:** (follows from the universal property of fiber products)

A morphism \( g : W \to X \times X \) in any scheme factors through \( \Delta : X \to X \times X \) if \( h \circ g = h' \circ g \)

\[ h|_U = h'|_U \Rightarrow (h,h'|_U) : U \to X \times X \text{ factors through } \Delta : (h,h'|_U) : U \to X \Delta \to X \times X \]

Image is \( T \)

the generic point of \( T \)

\( \Delta \) is a closed embedding and \( \Delta(X) \) contains the image of the generic point of \( T \) \( \Rightarrow \Delta(X) \supset \text{image of } T \)
\( \Rightarrow \Delta(x) \ni \text{image of the closed point of } T = \text{Spec } R \)

Claim (from previous page) \( \Rightarrow h(t_0) = h'(t_0) = : x_0 \)

\( x_1 := h(t_1) = h'(t_1) = \text{image of } U = \text{Spec } K \)

\( \Rightarrow x_0 \in \{x_1\} = : \mathbb{Z} \) with reduced induced scheme structure

and \( h(x_1) = 0_{\mathbb{Z}, x_1} \subset K \) given by \( h|_U = h'|_U \)

\( U = \text{Spec } K \rightarrow X \)

\( x_0 = h(t_0) = h'(t_0) \Rightarrow 0_{\mathbb{Z}, x_0} \subset R \) dominated.

Previous lemma \( \Rightarrow \nu = h': \text{Spec } R \rightarrow X \)

Conversely, suppose the valuative criterion is satisfied.

We show that \( \Delta(X) \subset X \times X \) is closed.

We use the following lemma:
Lemma: Let $f: X \rightarrow Y$ be a quasi-compact morphism of schemes. The subset $f(X)$ of $Y$ is closed iff it is closed under specialization, meaning $\forall y \in f(X)$, any specialization of $y$ also belongs to $f(X)$.

Proof: Hartshorne

Remark: In general any closed subset is closed under specialization.

Since $X$ is noetherian, $\Delta: X \rightarrow X \times X$ is quasi-compact. We show $\Delta(X)$ is closed under specialization.

Choose $\xi_1 \in \Delta(X) \subseteq X \times X$ and $\xi_0 \in \nu = \{ \xi_1 \}$ we show $\xi_0 \in \Delta(X)$. 


endon $\mathcal{Z}$ with the reduced induced scheme structure. $\xi$ is the generic point of $\mathcal{Z}$.

$$\mathcal{O}_{\mathcal{Z}, \xi} = K,$$ a field

$$= k(\xi)$$ residue field

$\mathcal{O}_{\mathcal{Z}, \xi_0} \subset K$

$\mathcal{O}_{\mathcal{Z}, \xi_0}$ is a valuation ring $R \subset K$ which dominates $\mathcal{O}_{\mathcal{Z}, \xi_0}$. Using the lemma from the previous lecture, we obtain a morphism

$$g : T := \text{Spec} R \longrightarrow X \times X$$

s.t.

$$g(t_1) = \xi_1, \quad g(t_0) = \xi_0$$

$h := \pi_1 \circ g, \quad h_0 := \pi_2 \circ g, \quad h|_U = h_1|_U \quad U = \text{Spec} K$

because $\xi_1 = g(t_1) \in \Delta(X)$

(see claim about $\Delta$)
So \((f_1 \circ g) \mid \Delta = (f_2 \circ g) \mid \Delta \implies f_1 \circ g = f_2 \circ g\).

Mutatis mutandis, \([g \text{ factors through } \Delta] \implies \exists g(t_0) \in \Delta(X).\]

Recall that properness is a substitute for compactness.

In a compact top. space, any closed subset is also compact, so any continuous map from a compact topological space is closed.

We use the condition of universally closed to define properness: being closed is not enough. \(A^1 \not\to \text{Spec } \mathbb{P}^1\).
Universally closed uses base change:

base change replaces the notion of extension of scalars:

\[ V \text{ vector space over } A : \rightarrow V \otimes_R E \text{ vector space over } R \]

Recall that tensor product geometrically correspond to fiber products.

We use fiber products to define base change.