Fibers of a morphism:

X a scheme, $x \in X$ $\quad \mathcal{O}_{X,x} \supset \mathfrak{m}_x$

Def: The residue field of X at x is

$$k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$$

Ex. II.2.7 The datum of a morphism from Spec K (K any field) to X with image x is equivalent to the datum of an inclusion $k(x) \hookrightarrow K$.

$\Rightarrow$ Given $x \in X$, we always have a morphism $\operatorname{Spec} k(x) \to X$ with image x, using the identity $k(x) \xrightarrow{=} k(x)$.

Def: Given a morphism of schemes $f: X \to Y$, let $y \in Y$ be a point, and $\operatorname{Spec} k(y) \hookrightarrow Y$ be the natural morphism as above.

The fiber X_y of f at y is

$$X_y := X \times_Y \operatorname{Spec} k(y), \text{ i.e.,}$$

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ \operatorname{Spec} k(y) \arrow[r, hook] & Y \end{tikzcd}$$

Ex. II.3.10: The underlying topological space of X_y is $f^{-1}(y) \subset X$.

This allows us to think of $f: X \to Y$ as a family of schemes $\{X_y, y \in Y\}$ parametrized by Y.

Def: Given a field k, a scheme X_0/k, (i.e., $X_0 \longrightarrow \operatorname{Spec} k$) a deformation of X_0 over a scheme Y over k is a k-morphism $X \longrightarrow Y$ (i.e.,
$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd}$$
) s.t. $\exists\, y_0 \in Y$ with $k(y_0) \cong k$ and $X_{y_0} \cong X_0$ as k-schemes.

The other fibers of f (at k-rational points, i.e., y s.t. $k(y) \cong k$) are called deformations of X_0.

Examples: (1) $X = \operatorname{Spec} k[x,y,t] \big/ (xy - t) \subset \mathbb{A}^3_k$

$$\begin{tikzcd} X = \operatorname{Spec} k[x,y,t]/(xy-t) \arrow[d] \arrow[r, phantom, "\subset"] & \mathbb{A}^3_k \arrow[dl] \\ Y = \operatorname{Spec} k[t] = \mathbb{A}^1_k & \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{A}^3_k = \operatorname{Spec} k[x,y,t] \arrow[d] & & k[x,y,t] \\ \mathbb{A}^1_k = \operatorname{Spec} k[t] & \Longleftrightarrow & k[t] \arrow[u, hook] \end{tikzcd}$$

Some fibers: $\quad t = a \in k \rightsquigarrow y \in Y$

$y \leftrightarrow (t-a) \subset k[t]$

$$X_y := \operatorname{Spec} k[x,y,t]/(xy-t) \times_Y \operatorname{Spec} k(y)$$

$$= \operatorname{Spec}\left(k[x,y,t]/(xy-t) \otimes_{k[t]} k(y) \right)$$

$$k(y) = \mathcal{O}_{Y,y}/m_y = k[t]_{(t-a)} / (t-a)_{(t-a)} \cong k.$$

$$0 \to (t-a) \to k[t] \to k[t]/(t-a) = \underset{\underset{k(y)}{\shortparallel}}{k} \to 0$$

localize!

$$0 \to (t-a)_{(t-a)} \to k[t]_{(t-a)} \to \underset{\underset{k(y).}{\shortparallel}}{k} \to 0$$

$$k[x,y,t]/(xy-t) \otimes_{k[t]} k[t]/(t-a)$$

$$0 \to (t-a) \to k[t] \to k[t]/(t-a) \to 0$$

~~tensor~~:

$$k[x,y,t]/(xy-t) \otimes_{k[t]} (t-a) \to k[x,y,t]/(xy-t) \otimes_{k[t]} k[t] \to$$

$$k[x,y,t]/(xy-t) \cong k[x,y,t]/(xy-t) \otimes_{k[t]} k[t]$$

$$\to k[x,y,t]/(xy-t) \otimes_{k[t]} k[t]/(t-a) \to 0$$

$$\Rightarrow k[x,y,t]/(xy-t) \otimes_{k[t]} k(y) \cong k[x,y,t]/(xy-t,t-a)$$

$$\cong k[x,y]/(xy-a)$$

$$\Rightarrow \quad X_y = \operatorname{Spec} k[x,y]/(xy-a) \quad \text{at } y=(t-a) \text{ "}=\text{" } a \in k$$

We have a family of hyperbolas X_y parametrized by $\mathbb{A}^1$, embedded in $\mathbb{A}^3$. When $a \neq 0$, $(xy-a)$ is prime $\Rightarrow X_y$ is irreducible (it is also reduced)

When $a=0$, then X_y is reducible, ideal is (xy)

$$X_y = \operatorname{Spec} k[x] \cup \operatorname{Spec} k[y] \subset \mathbb{A}^2 = \operatorname{Spec} k[x,y]$$

meet at one point.

$$= \operatorname{Spec} k[x,y]/(xy)$$

Similarly, when $X = \operatorname{Spec} k[x,y,t]/(ty-x^2) \subset \mathbb{A}^3_k$

we obtain a family of parabolas: $X_y = \operatorname{Spec} k[x,y]/(ay-x^2)$

When $a \neq 0$, X_y is an integral scheme.

When $a = 0$, $X_y = \operatorname{Spec} k[x]/(x^2)$ is non reduced.

- Separated and proper morphisms.

The Zariski topology is almost never Hausdorff.
(e.g. $\mathbb{A}^1_k$, k alg. closed).

Separated $\leftrightarrow$ Hausdorff

Proper $\leftrightarrow$ proper topologically

Idea: when a topology is Hausdorff, the diagonal is closed in the product topology on $X \times X$.

Def: Given $X \xrightarrow[\pi_X]{} S$, the diagonal morphism of X over S is the unique morphism

$\Delta : X \longrightarrow X \times_S X$ s.t. $p_1 \circ \Delta = p_2 \circ \Delta = \mathrm{Id}_X$,

i.e.,

$$\begin{tikzcd}
X \arrow[dr, "\exists!\Delta"'] \arrow[drr, bend left, "\mathrm{Id}_X"] \arrow[ddr, bend right, "\mathrm{Id}_X"'] & & \\
& X \times_S X \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "\pi_X"] \\
& X \arrow[r, "\pi_X"'] & S
\end{tikzcd}$$

<u>Definition</u>: A morphism $f : X \to Y$ is separated if the diagonal $\Delta : X \longrightarrow X \times_Y X$ is a closed embedding.

<u>Lemma</u>: Any morphism of affine schemes is separated.

Proof: $f : X \longrightarrow Y \qquad X = \operatorname{Spec} A, \ Y = \operatorname{Spec} B$

$\Longleftrightarrow \qquad f^\# : B \longrightarrow A$

$$X \times_Y X = \operatorname{Spec} A \otimes_B A$$

$$\begin{tikzcd}
A & & \\
& A \otimes_B A \arrow[ul, "\Delta^\#"'] & A \arrow[l, "a\otimes 1 \mapsfrom a"'] \arrow[ull, bend right, "\mathrm{Id}"'] \\
& A \arrow[u, "1\otimes a \mapsfrom a \in"] \arrow[uul, "\mathrm{Id}"] \arrow[ur, phantom, "\sqsupset"] & B \arrow[l, "f^\#"] \arrow[u, "f^\#"']
\end{tikzcd}$$

$\Delta^\#(1 \otimes a) = \Delta^\#(a \otimes 1) = a$

$\forall a \in A$

(uniqueness of $\Delta^\#$) $\Longrightarrow$

$$\Delta^\#(a \otimes a') = aa'$$

$$\Delta : \operatorname{Spec} A \longrightarrow \operatorname{Spec} A \otimes_B A$$

Δ is a closed embedding because $A \otimes_B A \longrightarrow\!\!\!\!\!\rightarrow A$ is surjective

$\square$

<u>Note</u>: the ideal of Δ in the affine case is

$$\mathrm{Ker}\left(A \otimes_B A \twoheadrightarrow A\right)$$

<u>Quintessential example of a non-separated morphism:</u>

k a field $\quad X = \mathrm{Spec}\, k[x]$, $Y = \mathrm{Spec}\, k[y]$ $\quad k[y]$

$$\begin{array}{ccc} X = \mathrm{Spec}\, k[x] & Y = \mathrm{Spec}\, k[y] & k[y] \\ \downarrow & \downarrow & \uparrow \\ \mathrm{Spec}\, k & \mathrm{Spec}\, k & k \end{array}$$

Glue X to Y as follows:

$U \subset X$: $\quad U = \mathrm{Spec}\, k[x, x^{-1}]$

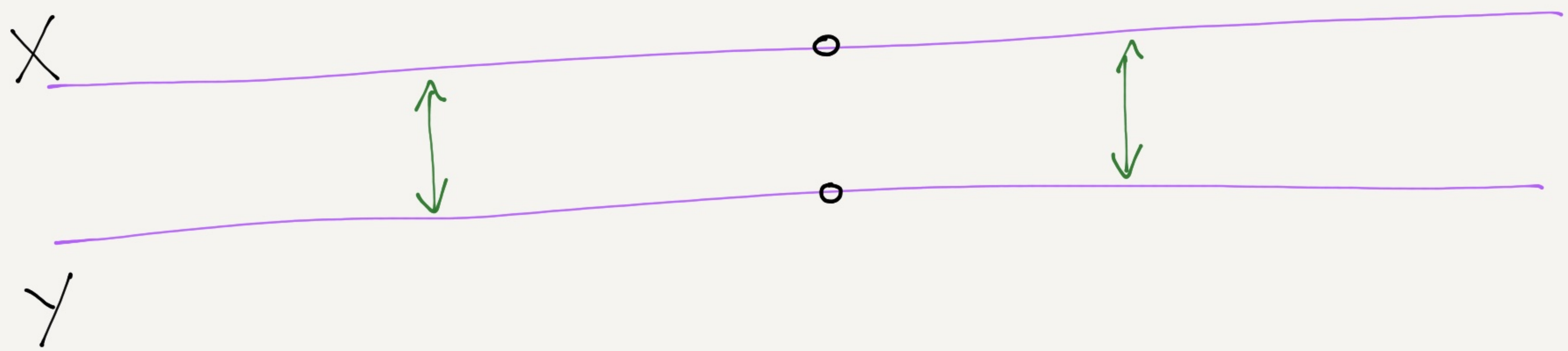
$V \subset Y$: $\quad V = \mathrm{Spec}\, k[y, y^{-1}]$

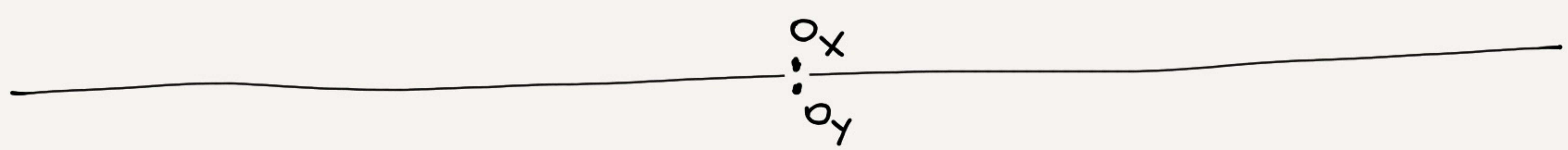
$$\varphi : U \xrightarrow{\cong} V \;,\; \varphi^\# : k[x, x^{-1}] \xleftarrow{\cong} k[y, y^{-1}]$$

$$x \longleftrightarrow y$$

$$x^{-1} \longleftrightarrow y^{-1}$$

Define $Z := X \underset{\varphi}{\cup} Y$ glued along φ. (see homework on glueing schemes)





These are non separated points (do not satisfy the valuative criterion).