Main examples for us: (for any ring $R$)

(1) $S = \{1, f, f^2, f^3, \dots\}$ for some $f \in R$

(2) $S = R \setminus \mathfrak{p}$ for some prime ideal $\mathfrak{p} \subset R$

For case (1): $S^{-1}R \underset{\text{notation}}{=} R[f^{-1}]$ which we have already seen

For case (2) we denote $S^{-1}R = R_{\mathfrak{p}}$

Lemma: $\forall\, R$, $\mathfrak{p} \in \operatorname{Spec} R$, $R_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ ($:=$ the ideal generated in $R_{\mathfrak{p}}$ by the image of $\mathfrak{p}$)

Idea of proof: Use a fact about local rings:
a ring $R$ is local with maximal ideal $\mathfrak{m}$ iff

$\forall \; x \in R \; , \; x$ is invertible $\Leftrightarrow x \notin \mathfrak{m}$.

Proposition: $\forall$ ring $R$, $\operatorname{Spec} R$ is a locally ringed space. In fact, $\forall\, \mathfrak{p} \in \operatorname{Spec} R$, $\mathcal{O}_{\mathfrak{p}} \cong R_{\mathfrak{p}}$.

The proof uses some lemmas:

Lemma 1: $\forall$ sheaf $\mathcal{F}$ on a topological space $X$, for any basis $\{U_i \mid i \in I\}$ of the topology of $X$,

$\forall x \in X$
$$\mathcal{F}_x \cong \varinjlim_{\substack{i \in I \\ U_i \ni x}} \mathcal{F}(U_i)$$

Lemma 2: $\forall$ ring $R$, $\forall\, \mathfrak{p} \in \operatorname{Spec} R$
$$\varinjlim_{f \notin \mathfrak{p}} R[f^{-1}] \cong R_{\mathfrak{p}}$$

<u>Note</u>: Lemma 1 & Lemma 2 $\Rightarrow$ Proposition:

$\mathfrak{p} \in \operatorname{Spec} R$ take the basis of the topology given by basic open sets: $\{ U_f \mid f \in R \}$

$$\Rightarrow \quad \mathcal{O}_{\mathfrak{p}} \underset{\text{lem. 1}}{=} \varinjlim_{\mathfrak{p} \in U_f} \mathcal{O}(U_f) = \varinjlim_{f \notin \mathfrak{p}} \mathcal{O}(U_f)$$

$$= \varinjlim_{f \notin \mathfrak{p}} R[f^{-1}] \underset{\text{lem. 2}}{=} R_{\mathfrak{p}}$$

<u>Proof of Lemma 1</u>: $\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U) \overset{?}{=} \varinjlim_{x \in U_i} \mathcal{F}(U_i)$

we have the natural inclusion

$$\begin{tikzcd}
\coprod_{x \in U_i} \mathcal{F}(U_i) \arrow[r, hook] \arrow[dr] & \coprod_{x \in U} \mathcal{F}(U) \arrow[d] \\
 & \mathcal{F}_x
\end{tikzcd}$$

We show that the composition factors through an injection (a) $\varinjlim_{\substack{i \in I \\ x \in U_i}} \mathcal{F}(U_i) \hookrightarrow \varinjlim_{x \in U} \mathcal{F}(U)$ and it is surjective (b)

(a) If $(U_i, s)$ and $(U_j, t)$ have the same image in $\varinjlim_{x \in U} \mathcal{F}(U)$, then $\exists \underset{\substack{\Psi \\ x}}{W} \subset U_i \cap U_j$ s.t.

$s|_W = t|_W$. $\exists\ k \in I$ s.t. $\underset{\substack{\Psi \\ x}}{U_k} \subset W$

$\Rightarrow \quad s|_{U_k} = t|_{U_k} \quad \Rightarrow$ $(U_i, s)$ and $(U_j, t)$ represent the same element of $\varinjlim_{x \in U_i} \mathcal{F}(U_i)$

(b) Given a pair $(U, s)$ representing an element of $\varinjlim_{x \in U} \mathcal{F}(U)$, $\exists\, i \in I$ s.t. $x \in U_i \subset U$,

then $(U, s) \sim (U_i, s|_{U_i})$ comes from $\varinjlim_{\substack{i \in I \\ x \in U_i}} \mathcal{F}(U_i)$

$\square$

<u>Proof of Lemma 2</u>: For any $f \notin \mathfrak{p}$, we have the natural map, obtained from the universal property of localizations:

$$\begin{tikzcd}
f \in R \arrow[r, "\varphi"] \arrow[d] & R_{\mathfrak{p}} \ni \frac{f}{1} \\
\frac{f}{1} \in R[f^{-1}] \arrow[ur, "S^{-1}\varphi"'] &
\end{tikzcd}$$

$S = \{1, f, f^2, \ldots\}$

$$\Rightarrow \quad \coprod_{f \notin p} R[f^{-1}] \longrightarrow R_p$$

we show that this induces an isomorphism

$$\varinjlim_{f \notin p} R[f^{-1}] \xrightarrow{\cong} R_p$$

(a) We determine the kernel of $\coprod R[f^{-1}] \longrightarrow R_p$.

Suppose $\frac{a}{f^n} \in R[f^{-1}]$ has image $\frac{0}{1} \in R_p$

this means $\exists\ s \in R \setminus p$ s.t. $s(a \cdot 1 - 0 \cdot f^n) = 0$

or $\underline{s \cdot a = 0}$ or $\frac{a}{1} \longmapsto \frac{0}{1}$ in $R[s^{-1}] \longrightarrow R[(sf)^{-1}]$

$$\boxed{\frac{a}{f^n} \in R[f^{-1}]}$$

claim $\frac{a}{f^n} \mapsto \frac{0}{1}$ in $R[(sf)^{-1}]$ because $asf = 0$ $\left(\frac{a}{1} \mapsto \frac{0}{1}\right)$

So $\frac{a}{f^n}$ represents 0 in the direct limit.

(b) Given $\frac{a}{f} \in R_p$ $f \notin p$

$\frac{a}{f}$ is the image of $\frac{a}{f} \in R[f^{-1}]$ $\square$.

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More generally, for any $R$-module $M$, we define a sheaf $\mathcal{M}$ of $\mathcal{O}$-modules on $\operatorname{Spec} R$ by setting

$$\mathcal{M}(U_f) := M[f^{-1}] := \text{localization of } M \text{ at } f$$
$$= M \times S / \sim$$
$$\text{or} \cong M \otimes_R R[f^{-1}]$$

For $U$ arbitrary open set

$$\mathcal{M}(U) := \varprojlim_{U_f \subset U} M[f^{-1}]$$

As in the case of $\mathcal{O}$, the stalks of $\mathcal{M}$ are

$$\mathcal{M}_{\mathfrak{p}} \cong \varinjlim_{f \notin \mathfrak{p}} \mathcal{M}(U_f) \cong M_{\mathfrak{p}} := \text{localization of } M \text{ at } R \setminus \mathfrak{p}$$
$$\cong M \otimes_R R_{\mathfrak{p}}$$

<u>Def</u>: A sheaf of $\mathcal{O}_R$-modules is called quasi-coherent if it is isomorphic, as an $\mathcal{O}_R$-module to $\mathcal{M}$ for some $R$-module $M$.

<u>Def</u>: <u>Morphisms of locally ringed spaces</u>:

(1) Given two local rings $(A, \mathfrak{m})$, $(B, \mathfrak{n})$ a hom. of rings $\varphi: A \to B$ is called local if $\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}$.

(2) A morphism of locally ringed spaces

$$\varphi: (Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$$

is a morphism of ringed spaces

$$\varphi : (Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$$

s.t. $\forall \; q \in Y$, the induced morphism

$$\mathcal{O}_{X,\varphi(q)} \longrightarrow \mathcal{O}_{Y,q}$$

is a local homomorphism of local rings.

<u>Note</u>: the construction of the induced morphism

$$\begin{array}{ccc} \mathcal{O}_{X,\varphi(q)} & \longrightarrow & \mathcal{O}_{Y,q} \\ \| & & \| \\ \varinjlim_{U \ni \varphi(q)} \mathcal{O}_X(U) & & \varinjlim_{V \ni q} \mathcal{O}_Y(V) \end{array}$$

$$\left(\varphi : Y \to X \,,\; \varphi^{\#} : \mathcal{O}_X \longrightarrow \varphi_* \mathcal{O}_Y\right) \rightsquigarrow \mathcal{O}_{X,\varphi(q)} \longrightarrow (\varphi_* \mathcal{O}_Y)_{\varphi(q)}$$

$$
\begin{tikzcd}
\varinjlim_{U\ni\varphi(q)} \mathcal{O}_X(U) \arrow[r] \arrow[d, equal] & \varinjlim_{U\ni\varphi(q)} \mathcal{O}_{\varphi_*\mathcal{O}_Y}(U) = \varinjlim_{U\ni\varphi(q)} \mathcal{O}_Y(\varphi^{-1}(U)) \\
\mathcal{O}_{X,\varphi(q)} \arrow[ddr] & = \varinjlim_{\varphi^{-1}(U)\ni q} \mathcal{O}_Y(\varphi^{-1}(U)) \arrow[d] \\
& \varinjlim_{V\ni q} \mathcal{O}_Y(V) \arrow[d, equal] \\
& \mathcal{O}_{Y,q}
\end{tikzcd}
$$

Def: An affine scheme is a locally ringed space which is isomorphic to the spectrum of a ring.

A scheme is a locally ringed space (or just a ringed space) which has a covering by open sets that are affine schemes.

## Understanding affine schemes a little better:

Note: the points of $\operatorname{Spec} R$ are not all closed!

In fact, given $p \in \operatorname{Spec} R$, let $\overline{\{p\}} \subset \operatorname{Spec} R$ be its closure. $\{p\} \subset \overline{\{p\}}$ is the smallest closed set containing $p$ $:=$ smallest $V(I)$ containing $p$

$=$ smallest $V(I)$ s.t. $I \subset p$

$= V(I)$ where $I$ is the largest ideal $\subset \mathfrak{p}$.

$= V(\mathfrak{p}) = \{ \mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \supset \mathfrak{p} \}$

$\{\mathfrak{p}\}$ is closed $\iff \{\mathfrak{p}\} = \overline{\{\mathfrak{p}\}} = V(\mathfrak{p}) = \{ \mathfrak{q} \mid \mathfrak{q} \supset \mathfrak{p} \}$

$\iff \mathfrak{p}$ maximal.