Def: Given a collection of maps \( \phi_{ij} : S_j \rightarrow S_i \)
\( \forall i \leq j \) s.t. \( \forall i \leq j \leq k \) \( \phi_{ik} = \phi_{ij} \circ \phi_{jk} \)
we define the inverse limit \( \lim_{i \in I} S_i \) to be

\[
\lim_{i \in I} S_i := \left\{ (x_i)_{i \in I} \mid x_i \in S_i \quad \forall i \leq j \quad \phi_{ij}(x_j) = x_i \right\}
\]

\( \bigcap_{i \in I} \bigcup_{i \in I} S_i \)

(strictly speaking, we do not need a directed set)
here, only a partially ordered set
As we saw, for \( U \subseteq \text{Spec } R \), we define:

\[
\text{Def: } G(U) := \text{lim } R[f^{-1}]
\]

\[ U \subseteq U_f \]

Note: \( f, g \in R \) what does it mean for \( U_f \) to be contained in \( U_g \)?

\[ U_f \subseteq U_g \implies V(f) \supseteq V(g) \]

\[
\{ \mu \in \text{Spec } R \mid f \in \mu \} \supseteq \{ \mu \in \text{Spec } R \mid g \in \mu \}
\]

\[
\begin{align*}
\forall f \in \text{Spec } R, \quad & \forall g \in \text{Spec } R, \quad p \in g \implies p \in f \\
= \quad & \forall f \in \text{Spec } R, \quad p \in g \implies p \in f \\
\end{align*}
\]

\[
\bigcap_{p \in f} p = \sqrt{f} \subseteq \bigcap_{p \in g} p = \sqrt{g}
\]
(\Rightarrow) \quad f \in \sqrt{\langle g \rangle}

(\Leftarrow) \quad \exists \ n \text{ s.t. } g \mid f^n

---

Exercise: Verify that the definition above for \( O(U) \) indeed defines a sheaf on \( \text{Spec} R \). (Also see “Geometry of Schemes”)

We are going to see this is a sheaf in a different way, using the “space étale”.

Def: The stalk of a presheaf \( \mathcal{F} \) on \( X \) at a point \( x \in X \) (for any topological space \( X \)) is

\[ \mathcal{F}_x := \lim_{x \in U \subseteq X} \mathcal{F}(U) \]
More concretely: 

\[ \mathcal{F}_x := \{ \mathcal{F}(U) \mid x \in U \cap X \} / \sim \]

\[ = \{ (U, s) \mid x \in U \cap X, s \in \mathcal{F}(U) \} / \sim \]

where \((U, s) \sim (V, t) \iff \exists \ W \subset U \cup V \ s.t. \ s|_W = t|_W\)

**Terminology:** For \( U \subset X \) and \( s \in \mathcal{F}(U) \) and \( x \in U \), we call the image of \((U, s)\) in \( \mathcal{F}_x \) the germ of \( s \) at \( x \) and we denote it \( s(x) \).

**Def:** The sheaf \( \mathcal{F}_x \) associated to a presheaf \( \mathcal{F} \) is the unique (up to isomorphism) sheaf with a morphism.
\( f : \mathcal{F} \rightarrow \mathcal{G} \) s.t., for any sheaf \( \mathcal{G} \), any morphism of sheaves \( \mathcal{F} \rightarrow \mathcal{G} \) factors uniquely through \( \mathcal{F} \\)

\[ \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \]

\( \exists ! \tilde{\varphi} \) which makes the diagram commute.

**Def:** A morphism of sheaves \( \varphi : \mathcal{F} \rightarrow \mathcal{G} \) is the data of maps \( \varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U) \) for all open sets \( U \) such that the following diagram commutes:

\[ \mathcal{F}(U) \xrightarrow{\varphi(U)} \mathcal{G}(U) \]

restriction

\[ \mathcal{F}(V) \xrightarrow{\varphi(V)} \mathcal{G}(V) \]

restriction
A morphism of sheaves is a morphism of the underlying presheaves.

We construct the sheaf associated to a presheaf using the "espace étale" of a presheaf.

**Def:** The espace étale of a presheaf $\mathcal{F}$ on $X$ is the set $\mathcal{E}_X := \prod_{x \in X} \mathcal{F}_x$ with a topology defined below.

Given $U \subseteq X$ any $s \in \mathcal{F}(U)$ defines a section of $\pi$:

$$s : U \to \prod_{x \in U} \mathcal{F}_x \subseteq \mathcal{E}_X \quad x \mapsto s(x) \in \mathcal{F}_x$$
We endow $\mathcal{F}$ with the topology whose open sets are unions of sets of the form $s(U) \subseteq \mathcal{F}$.

**Def:** The sheaf $\mathcal{F}$ can be defined as the sheaf of continuous sections of $\pi_1$ for the above topology.

More concretely: $U \subset X$ open, we describe $\tilde{\mathcal{F}}(U)$.

$$
\tilde{\mathcal{F}}(U) := \{ f : U \rightarrow \mathcal{F} \mid f \text{ continuous and } T\{f=\text{Id}_U\} \}
$$

$$
= \{ f : U \rightarrow \mathcal{F} \mid f \text{ continuous and } \forall x \in U, f(x) \in \mathcal{F}_x \}
$$

Understanding the continuity: fix $U \subset X$

$f : U \rightarrow \mathcal{F}$ continuous if $\forall V \subset \mathcal{F}$ open,

$f^{-1}(V) \subset U$ is open.
WLOG we can assume $V = s(W)$ for some $W \subset U$ open and $s \in \mathfrak{F}(W)$. What does it mean to say $f^{-1}(s(W))$ is open in $U$?

\[
f^{-1}(s(W)) = \left\{ x \in U \mid f(x) \in s(W) \right\}
= \left\{ x \in U \mid f(x) = s(x) \right\}
= s^{-1}(f(W))
\]