<u>Example:</u> $Y \subset \mathbb{A}^n$ a hypersurface.

by def., $\exists\, f \in A = k[y_1, \dots, y_n]$ s.t. $Y = Z(f) = Z(\langle f \rangle)$

Then $I(Y) = \sqrt{\langle f \rangle}$ by Nullstellensatz

$\langle f \rangle$ = ideal generated by f.

We can write $f = g_1^{r_1} g_2^{r_2} \cdots g_m^{r_m}$ where g_i are irreducible polynomials and $r_i > 0$ and g_i do not divide each other.

Then $\sqrt{\langle f \rangle} = \langle g_1 \cdots g_m \rangle$

Recall $\sqrt{I} = \bigcap\limits_{p \supset I} p$ for any ideal

The set of prime ideals containing f is $\{\langle g_1 \rangle, \dots, \langle g_m \rangle\}$

$$\sqrt{\langle f \rangle} = \bigcap_{i=1}^{m} \langle g_i \rangle = \langle g_1 \cdots g_m \rangle$$

So $$Z(f) = \bigcup_{i=1}^{m} Z(g_i)$$

irreducible closed in $\mathbb{A}^n$

$$Z(g_i) \not\subset Z(g_j) \text{ for } i \neq j$$

because $\langle g_j \rangle \not\subset \langle g_i \rangle$

So the $Z(g_j)$ are the irreducible components of

$$Y = Z(f).$$

Y is irreducible if $m=1$, meaning f is a prime power.

$$f = g_1^{r_1}, \quad I(Y) = \langle g_1 \rangle$$

$$\dim Y = \dim A(Y) = \dim A/\langle g_1 \rangle$$

$$= \dim A - \operatorname{height}\langle g_1 \rangle$$

Prop. 1.11A: For any $f \in A$ noetherian ring, if f is not a zero divisor, then the height of any minimal prime containing f is 1.

$$\Rightarrow \quad \text{height}\langle g, \rangle = 1$$

and $\dim Y = \dim A - 1 = n-1$.

A few words about cones (good sources of examples): also further link projective to affine

The affine cone over a projective variety:

Let $\theta : \mathbb{A}^{n+1} \setminus \{0\} \twoheadrightarrow \mathbb{P}^n$ be the quotient map.

For any projective variety $Y \subset \mathbb{P}^n$, define affine cone over Y to be $C(Y) := \theta^{-1}(Y) \cup \{0\} \subset \mathbb{A}^{n+1}$

By definition the ideal of $C(Y)$ as an affine variety is equal to the homogeneous ideal of Y as a projective variety:

$$
\begin{array}{ccc}
I(C(Y)) & = & I(Y) \\
\cap & & \cap \\
k[x_0, \dots, x_n] & = & k[x_0, \dots, x_n] = S \\
\text{coordinate ring of } \mathbb{A}^{n+1} & & \text{coordinate ring of } \mathbb{P}^n
\end{array}
$$

<u>The projective cone over a projective variety</u>:

Y and $C(Y)$ as above. Choose an embedding

$\mathbb{A}^{n+1} \hookrightarrow \mathbb{P}^{n+1}$ e.g. as $U_{n+1} = \mathbb{P}^{n+1} \setminus Z(x_{n+1})$

$(b_0, \dots, b_n) \longmapsto (b_0, \dots, b_n, 1)$

The projective cone $\overline{C(Y)}$ of Y is the closure of $C(Y)$ in $\mathbb{P}^{n+1}$ via an embedding as above.

$$\mathbb{A}^{n+1} \overset{\sim}{\hookrightarrow} U_{n+1} \subset \mathbb{P}^{n+1}$$

The ideal of $\overline{C(Y)}$ in $\mathbb{P}^{n+1}$ is the homogenization of $I(C(Y))$ in $k[x_0, \dots, x_{n+1}]$ with respect to x_{n+1}.

Sheaves

Presheaves: **Def**: Let X be a topological space.

A presheaf $\mathcal{F}$ (of sets) on X is the data,

(1) for each open set $U \subset X$, of a set $\mathcal{F}(U)$,

(2) for every inclusion of open sets $V \subset U \subset X$, of

a restriction map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$,

such that:

(1) $\mathcal{F}(\emptyset) = \{\emptyset\}$ the set with one element,

(2) $\rho_{UU} = Id_U$ the identity map of $\mathcal{F}(U)$,

(3) for all inclusions of open sets $W \subset V \subset U$, we have $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\rho_{UV}"] \arrow[rr, bend right, "\rho_{UW}"'] & \mathcal{F}(V) \arrow[r, "\rho_{VW}"] & \mathcal{F}(W) \end{tikzcd}$$

<u>Note</u>: A presheaf is a contravariant functor from the category of open sets of X to the category of sets.

The target category can also be, abelian groups, rings, modules over a fixed ring, etc. (maps of sets would have to be changed to morphisms in each target category), $\{\phi\}$ would have to be replaced by a final object.

<u>Example</u>: Constant presheaves: $\mathcal{F}(U) = S$ for a fixed set S, $\forall\ U \neq \phi$.

<u>Notation</u>: From now on, we denote $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ by $|_V := \rho_{UV}$.

<u>Definition</u>: A presheaf $\mathcal{F}$ on X is a sheaf if it satisfies the following.

(1) for all open sets U and all open coverings

$U = \bigcup_{i \in I} V_i$, and, for all $s, t \in \mathcal{F}(U)$ "sections"

of $\mathcal{F}$ over U, if $s|_{V_i} = t|_{V_i}$ $\forall i \in I$, then $s = t$.

(2) for all U and all $U = \bigcup_{i \in I} V_i$, and, all

collections $\{ s_i \in \mathcal{F}(V_i) \mid i \in I \}$ s.t. $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$

$\forall i, j$,

$\exists \; s \in \mathcal{F}(U)$ s.t. $s|_{V_i} = s_i \;\; \forall i$.

<u>Note</u>: the constant presheaf is most often not a sheaf.

<u>Examples of sheaves</u>:

(1) X, S two topological spaces.

$\mathcal{F} :=$ sheaf of continuous functions from X to S

$\mathcal{F}(U) := \{$ continuous functions from U to $S \}$

If S is given the discrete topology, then $\mathcal{F}$ is the constant "sheaf" on X with values in S.

(2) Sheaves of C^∞ functions or differential forms on C^∞ manifolds.

(3) sheaves of holomorphic functions on complex analytic spaces, or real analytic functions on real analytic spaces.

(4) $\pi : E \twoheadrightarrow X$ continuous. (surjective)

sheaf of continuous sections of π:

$$\mathcal{F}(U) := \{ s : U \to E \text{ cont.} \mid \pi \circ s = \mathrm{Id}_U \}$$

For k alg. closed, by Nullstellensatz, every maximal ideal in $k[y_1, \dots, y_n]$ is of the form

$$(y_1 - b_1, \dots, y_n - b_n) \text{ for some } (b_1, \dots, b_n) \in k^n.$$

In other words we have a bijection

between $k^n = \{ (b_1, \dots, b_n) \mid b_i \in k \}$ and the set of maximal ideals of $A = k[y_1, \dots, y_n]$.

In general, we can always embed

$$k^n \subset \{\text{maximal ideals}\}$$

So, as a first step, we replace affine space with the set of maximal ideals of $k[y_1, \dots, y_n]$