Recall that $\mathbb{P}^n = \bigcup_{i=0}^{n} U_i$ $\quad U_i := \operatorname{Im} \varphi_i$

$\varphi_i : \mathbb{A}^n \hookrightarrow \mathbb{P}^n$

$(t_1, \ldots, t_n) \longmapsto (t_1, \ldots, \underset{\text{i-th coord.}}{1}, \ldots, t_n)$

$U_i = \mathbb{P}^n \setminus Z(x_i)$

Lemma: For $Y \subset \mathbb{P}^n$ projective, irred., $\forall i \in \{0, \ldots, n\}$ if $Y \cap U_i \neq \emptyset$, then $\dim Y = \dim (Y \cap U_i)$

Proof: WLOG assume $i = 0$.

By the theorem,
$$\begin{aligned}
\dim Y &= \dim S(Y) - 1 \\
&= \dim S - \operatorname{height} I(Y) - 1 \\
&= \operatorname{tdeg}_k k(x_0, \ldots, x_n) - \operatorname{height} I(Y) - 1 \\
&= n + 1 - \operatorname{height} I(Y) - 1 \\
&= n - \operatorname{height} I(Y)
\end{aligned}$$

Similarly $Y \cap U_0 \subset U_0 \underset{\varphi_0}{\cong} \mathbb{A}^n$ homeomorphism

By the theorem, $\dim(Y \cap U_0) = \dim A(Y \cap U_0)$

$$= \dim A - \text{height}\, I(Y \cap U_0)$$

$$= \text{trdeg}_k k(y_1, \dots, y_n) - \text{height}\, I(Y \cap U_0)$$

$$= n - \text{height}\, I(Y \cap U_0)$$

So $\dim Y = \dim(Y \cap U_0) \iff \text{height}\, I(Y) = \text{height}\, I(Y \cap U_0)$

We have $I(Y) \subset S = k[x_0, \dots, x_n]$

$$I(Y \cap U_0) \subset A = k[y_1, \dots, y_n]$$

$$\varphi_0 : \mathbb{A}^n \xrightarrow{\cong} U_0 \subset \mathbb{P}^n$$

$$(b_1, \dots, b_n) \longmapsto (1, b_1, \dots, b_n)$$

$$\left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right) \longleftrightarrow (a_0, \dots, a_n) \quad a_0 \neq 0$$

We have $h : A \longrightarrow S$ homogenization

$$P \longmapsto P_h := x_0^d \, P\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$$

$$P(y_1, \dots, y_n) = P \qquad d := \deg P$$

and dehomogenization $dh : S \longrightarrow A$

$$\begin{array}{ccc} Q & \longmapsto & Q_{dh} := Q(1, y_1, \dots, y_n) \\ \shortparallel & & \\ Q(x_0, \dots, x_n) & & \end{array}$$

<u>Check</u>: $I(Y)_{dh} = I(Y \cap U_0)$, $I(Y) = I(Y \cap U_0)_h$

true because $Y \cap U_0 \neq \phi$ and Y is irreducible.

where $I(Y)_{dh}$:= ideal generated by the dehomogenization of all the elements of $I(Y)$

$I(Y \cap U_0)_h$:= ideal generated by the homogenization of all the elements of $I(Y \cap U_0)$

To show that $I(Y)$ and $I(Y \cap U_0)$ have the same height, we show that we have a bijection between their sets of chains of prime ideals.

Given $\quad I(Y \cap U_0) \supseteq \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_m$

homogenize to get $\quad I(Y) \supseteq \mathfrak{p}_{0,h} \supsetneq \mathfrak{p}_{1,h} \supsetneq \cdots \supsetneq \mathfrak{p}_{m,h}$

the inclusions are strict because $\quad (P_h)_{dh} = P$

$$\Rightarrow \quad (\mathfrak{p}_{i,h})_{dh} = \mathfrak{p}_i$$

$$\Rightarrow \quad \text{height } I(Y) \geq \text{height } I(Y \cap U_0)$$

Now start with $\quad I(Y) \supseteq \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_m$

dehomogenize to get $\quad I(Y \cap U_0) \supseteq \mathfrak{p}_{0,dh} \supseteq \mathfrak{p}_{1,dh} \supseteq \cdots \supseteq \mathfrak{p}_{m,dh}$

need to show that the inclusions are strict.

$$(Q_{dh})_h = x_0^r Q \quad \text{for some } r \leq 0$$

e.g. $\quad x_0^2 x_1 + x_0 x_3^2 \overset{dh}{\rightsquigarrow} y_1 + y_3^2 \overset{h}{\rightsquigarrow} x_0 x_1 + x_3^2$

For $j \in \{0, \dots, n-1\}$, we show $\mathfrak{p}_{j,dh} \supsetneq \mathfrak{p}_{j+1,dh}$

we know $\mathfrak{p}_j \supsetneq \mathfrak{p}_{j+1}$, choose $a \in \mathfrak{p}_j \setminus \mathfrak{p}_{j+1}$
a homogeneous.

if $a_{dh} \in \mathfrak{p}_{j+1,dh}$, then $\exists\ b \in \mathfrak{p}_{j+1}$ s.t. $a_{dh} = b_{dh}$

$\Rightarrow \exists r > 0$ s.t. $a = x_0^r b$ or $b = x_0^r a \in \mathfrak{p}_{j+1}$

$a = x_0^r b \Downarrow a \notin \mathfrak{p}_{j+1}$, $a \notin \mathfrak{p}_{j+1}$ not allowed

$b = x_0^r a \in \mathfrak{p}_{j+1}$, $a \notin \mathfrak{p}_{j+1}$ $\Downarrow$ $x_0 \in \mathfrak{p}_{j+1} \subset I(Y)$ $\Downarrow$

$x_0 \in I(Y) \Rightarrow Y \subset Z(x_0)$

$\Rightarrow Y \cap U_0 = \emptyset$

not allowed.

So all the inclusions are strict, and

height $I(Y) \leq$ height $I(Y \cap U_0)$ $\square$

Lemma: A subset $Y \neq \emptyset$ of a topological space X is irreducible iff $\forall\ Y_1, Y_2$ closed in X,

$$Y \subset Y_1 \cup Y_2 \implies Y \subset Y_1 \text{ or } Y \subset Y_2$$

(Y is given the topology induced from X)

Proof: exercise.

Proposition: In a noetherian topological space X, any closed subset $Y \neq \emptyset$ is a finite union $Y_1 \cup \cdots \cup Y_r$ of irreducible closed subsets. If $\forall\ i \neq j$, $Y_i \not\subset Y_j$, then the decomposition is unique up to reindexing of the Y_i.

Definition: The Y_i as above are called the irreducible components of Y.

<u>Proof:</u> <u>Existence</u> Part

$\mathfrak{S} := \{$non-empty closed subsets that are NOT finite unions of irreducible closed subsets$\}$

If $\mathfrak{S} \neq \phi$, then, because X is noetherian, every descending chain of elements of $\mathfrak{S}$ has a minimal element.
So, by Zorn's lemma, $\mathfrak{S}$ has a minimal element, say Y.

$Y \in \mathfrak{S} \implies Y$ is not irreducible

$\implies \exists\ Y_1, Y_2$ closed, not empty in Y s.t.

$Y = Y_1 \cup Y_2$, $Y_1 \subsetneq Y$ and $Y_2 \subsetneq Y$

Since Y is minimal in $\mathfrak{S}$, Y_1 and $Y_2 \notin \mathfrak{S}$.

$\implies Y_i =$ finite union of irreducible closed subsets

$\Rightarrow$ Y = finite union of irred. closed subsets

$\Rightarrow$ $Y \notin \mathfrak{S}$ contradiction.

Uniqueness up to reindexing:

Suppose $Y = Y_1 \cup \cdots \cup Y_r = Y_1' \cup \cdots \cup Y_s'$

$Y_i \not\subset Y_j$, $i \neq j$ $\qquad$ $Y_i' \not\subset Y_j'$, $i \neq j$

Y_r is irreducible and contained in $Y_1' \cup \cdots \cup Y_s'$,

by the lemma, $\exists\, j$ s.t. $Y_r \subset Y_j'$

similarly, $\exists\, i$ s.t. $Y_j' \subset Y_i$ $\Rightarrow$ $Y_r \subset Y_i$

$\Rightarrow$ $i = r$ and $Y_j' \subset Y_r$ $\Rightarrow$ $Y_j' = Y_r$

Remove Y_r from both sides and reindex to obtain

$Y_1 \cup \cdots \cup Y_{r-1} = Y_1' \cup \cdots \cup Y_{s-1}'$

repeat to obtain $r = s$ and $\forall i, \exists j$ s.t. $Y_i = Y'_j$.

$\square$

Remark (Tammy): we didn't need Zorn's lemma:

If $\mathfrak{S}$ does not have a minimal element, then we can produce an infinite descending chain of closed subsets of X.