Def: The dimension of a topological space $Y$ is

\[ \dim Y := \sup \{ n \mid \exists \text{ chain of distinct irreducible closed subsets } Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_n \} \]

Example: \( \dim \mathbb{A}^n > n \): chain

\[ k^n = Y_0 \supsetneq Y_1 \supsetneq Y_2 \supsetneq \cdots \supsetneq Y_n \]

\[ \mathbb{Z}(y_n) \supsetneq \mathbb{Z}(y_n, y_{n-1}) \]

Meaning:

Theorem: (1) If $Y \subset \mathbb{A}^n$ is an irreducible affine variety, then

\[ \dim Y = \dim A(Y) = \text{str deg } \text{Frac} \left( \frac{A(Y)}{(x)} \right) \]

(2) If $Y \subset \mathbb{P}^n$ is an irreducible projective variety, then

\[ \dim Y = \dim S(Y) - 1 \]
Preliminaries for the theorem:

**Lemma:** \( Y \subset A^n \) or \( P^n \) affine or projective variety.

Then \( Y \) irreducible \( \iff \) \( I(Y) \) is prime.

**Proof:** Assume \( I(Y) \) is prime.

If \( Y = Y_1 \cup Y_2 \), \( Y_1 \neq \emptyset \), \( Y_2 \neq \emptyset \), \( Y_1, Y_2 \) closed.

Then \[ I(Y) = I(Y_1) \cap I(Y_2) \cup I(Y_1) \cdot I(Y_2) \]

\[ \Rightarrow I(Y_1) \cdot I(Y_2) \subset I(Y) \]

\[ \Rightarrow I(Y_1) \subset I(Y) \cap I(Y_2) \subset I(Y) \]

because \( I(Y) \) is prime.
general fact for affine or projective varieties:

\[ Y \subset Z \implies I(Z) \subset I(Y) \]

So, since \( Y_1, Y_2 \subset Y \), we have \( I(Y) \subset I(Y_1) \) and \( I(Y) \subset I(Y_2) \)

So from the previous page:

\[ I(Y_1) = I(Y) \cap I(Y_2) = I(Y) \]

\[ \implies Y_1 = Y \cap Y_2 = Y. \]

Now assume \( Y \) is irreducible.

\( \text{If } ab \in I(Y), \text{ then } Z(ab) \supseteq Z(I(Y)) = Y \)

\[ Z(a) \cup Z(b) \supseteq Z(a \cap b) \]

\[ \implies (Z(a) \cap Y) \cup (Z(b) \cap Y) = Y \]
\[ \forall \text{ irreducible } \implies Z(a) \cap Y = Y \cap Z(b) \cap Y = Y \]
\[ = Z(a) \supset Y \cap Z(b) \supset Y \]
\[ = a \in I(Y) \cap b \in I(Y). \]

**Definition:**

<table>
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<th><strong>Dimension of a commutative rig</strong> (all rings have identity elements, denoted (1))</th>
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<tr>
<td>The height of (p) is</td>
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<td>(\text{height}(p) := \sup { n \mid \exists p_0, p_1, \ldots, p_n \subseteq p } )</td>
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<td>The Krull dimension of (R) is</td>
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<td>(\text{dim}(R) := \sup { \text{height}(p) \mid p \subset R \text{ prime} } )</td>
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\[ \text{dim} (R) = \inf \left\{ n \mid \exists \ p_0 \supsetneq \cdots \supsetneq p_n \subset R \right\} \]

Prop (1.8A): If an integral domain which is also a finitely generated \( k \)-algebra, then
\[ \text{dim} (R/\mathfrak{p}) + \text{height} (\mathfrak{p}) = \text{dim} R \]

\( \forall \) prime ideals \( \mathfrak{p} \subset R \).

Also \( \text{dim} R = \text{tideg} \text{Frac}(R) \), also denote \( K(R) := \text{Frac}(R) \)

Recall: for any extension of fields \( K \subset L \),
\[ \text{tideg} L = \sup \left\{ n \mid \exists \ x_1, \ldots, x_n \in L \text{ algebraically independent over } K \right\} \]
\[ A = k[y_1, \ldots, y_n], \text{ then } K(A) = k(y_1, \ldots, y_n) \]
\[ \text{tr.deg}_k K(A) = n \]

\[ A(Y) = k[x, y] / (y-x^2), \text{ if } Y = \text{parabola} = \mathbb{Z}(y-x^2) \]
\[ \cong k[x] \]
\[ K(A(Y)) = \text{Frac}(k[x, y] / (y-x^2)) \]
\[ \cong k(x) \]
\[ \text{tr.deg}_k K(A(Y)) = 1. \]

Proof of the theorem: First note that, by the lemma, \( Y \) irreducible \( \iff \) \( I(Y) \) prime
\( \iff \) \( A(Y) = A / I(Y) \) is an integral domain.
Proof of \( \dim Y = \dim A(Y) \) for \( Y \) affine.

The data of a chain of irreducible closed subsets of \( Y : Y \supseteq Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n = \emptyset \) is equivalent (by Nullstellensatz) to the data of the chain of prime ideals

\[
\mathfrak{P}(Y) \supseteq \mathfrak{P}(Y_0) \supseteq \mathfrak{P}(Y_1) \supseteq \cdots \supseteq \mathfrak{P}(Y_n) \supseteq A = k[y_1, \ldots, y_n] \]

\( \iff \)

\[
(0) \subseteq \frac{\mathfrak{P}(Y_0)}{\mathfrak{P}(Y)} \supseteq \frac{\mathfrak{P}(Y_1)}{\mathfrak{P}(Y)} \supseteq \cdots \supseteq \frac{\mathfrak{P}(Y_n)}{\mathfrak{P}(Y)} \supseteq \frac{A}{\mathfrak{P}(Y)} = A(Y) \]

\( \implies \)

\( \dim Y = \dim A(Y) \)
Proof of $\dim Y = \dim S(Y) - 1$ for $Y$ irreducible projective.

The data of a chain of irreducible closed subsets of $Y$:

$Y \supseteq Y_0 \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_n \neq \emptyset$

is equivalent to the data of their homogeneous prime ideals (by homogeneous Nullstellensatz):

$I(Y) \subseteq I(Y_0) \supsetneq I(Y_1) \supsetneq \cdots \supsetneq I(Y_n) \supsetneq S \supsetneq S(Y) \supsetneq S(Y+)$

$\Rightarrow \quad 0 = \frac{I(Y_0)}{I(Y)} \supsetneq \frac{I(Y_1)}{I(Y)} \supsetneq \cdots \supsetneq \frac{I(Y_n)}{I(Y)} \supsetneq \frac{I(Y)}{I(Y)}$

$\Rightarrow \quad \dim Y = \text{height } S(Y) + 1 - 1 = \dim S(Y) - \dim \sqrt{S(Y)} - 1$. 

\[
\frac{S(Y)}{S(Y)^+} = \frac{S/I(Y)}{S/I(Y)} \cong = \frac{S}{S_I} \cong k
\]

\[\dim Y = \dim S(Y) - \text{NullDim}(k) = \dim S(Y)\]

\[\text{NullDim}(k) = 0\]

because \( k \) has exactly one prime ideal: \((0) \subset k\).

\[A(k) = A(\mathbb{A}^1) = k[y,]\]

\[1 - \dim \mathbb{A}^1 = \text{NullDim} k[y,]\]