Def: The dimension of a topological space Y is

$$\dim Y := \operatorname{Sup}\{n \mid \exists \text{ chain of distinct irreducible closed subsets } Y_0 \supsetneq Y_1 \supsetneq Y_2 \supsetneq \cdots \supsetneq Y_n\}$$

$Y_0 \subset Y$

Example: $\dim k^n \geq n$: chain

$$k^n = Y_0 \supsetneq \underset{Z(y_n)}{\underset{\parallel}{Y_1}} \supsetneq \underset{Z(y_n, y_{n-1})}{\underset{\parallel}{Y_2}} \supsetneq \cdots \supsetneq \underset{Y_n}{\underset{\parallel}{\{0\}}} = Z(y_1, \ldots, y_n)$$

Main goal:

Theorem: (1) If $Y \subset \mathbb{A}^n$ is an irreducible affine variety, then

$$\dim Y = \dim A(Y) = \operatorname{trdeg}_k \operatorname{Frac}(A(Y))$$

(2) If $Y \subset \mathbb{P}^n$ is an irreducible projective variety, then $\dim Y = \dim S(Y) - 1$.

Preliminaries for the theorem:

Lemma: $Y \subset \mathbb{A}^n$ or $\mathbb{P}^n$ affine or projective variety.

Then Y irreducible $\Longleftrightarrow$ $I(Y)$ is prime.

Proof: Assume $I(Y)$ is prime.

If $Y = Y_1 \cup Y_2$, $Y_1 \neq \phi$, $Y_2 \neq \phi$, Y_1, Y_2 closed.

Then
$$I(Y) = I(Y_1) \cap I(Y_2)$$
$$\cup$$
$$I(Y_1) \cdot I(Y_2)$$

$$\Rightarrow I(Y_1) \cdot I(Y_2) \subset I(Y)$$
$$\Rightarrow I(Y_1) \subseteq I(Y) \text{ or } I(Y_2) \subseteq I(Y)$$
because $I(Y)$ is prime

general fact for affine or projective varieties.

$$Y \subset Z \implies I(Z) \subset I(Y)$$

So, since $Y_1, Y_2 \subset Y$, we have $I(Y) \subset I(Y_1)$ and $I(Y) \subset I(Y_2)$

So from the previous page:

$$I(Y_1) = I(Y) \text{ or } I(Y_2) = I(Y)$$

$$\implies Y_1 = Y \text{ or } Y_2 = Y.$$

Now assume Y is irreducible.

If $ab \in I(Y)$, then $Z(ab) \supset Z(I(Y)) = Y$

$$Z(ab) = Z(a) \cup Z(b)$$

$$\implies (Z(a) \cap Y) \cup (Z(b) \cap Y) = Y$$

Y irreducible $\Rightarrow$ $Z(a) \cap Y = Y$ or $Z(b) \cap Y = Y$

$\Rightarrow$ $Z(a) \supset Y$ or $Z(b) \supset Y$

$\Rightarrow$ $a \in I(Y)$ or $b \in I(Y)$. $\square$

Dimension of a commutative ring (all rings have identity elements, denoted 1)

Def: R a comm. ring. $\mathfrak{p} \subset R$ prime ideal

(1) The height of $\mathfrak{p}$ is

$$\text{height}(\mathfrak{p}) := \sup \{ n \mid \exists\ \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subseteq \mathfrak{p} \}$$

$\mathfrak{p}_i$ all prime

(2) The Krull dimension of R is

$$\dim(R) := \sup \{ \text{height}(\mathfrak{p}) \mid \mathfrak{p} \subset R \text{ prime} \}$$

$$\dim(R) = \sup\{n \mid \exists\ \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subsetneq R\}$$
$\mathfrak{p}_i$ prime

<u>Prop (1.8A)</u> : R an integral domain which is also a finitely generated k-algebra, then

$$\dim(R/\mathfrak{p}) + \operatorname{height}(\mathfrak{p}) = \dim R$$

$\forall$ prime ideals $\mathfrak{p} \subset R$.

Also $\dim R = \operatorname{trdeg}_k \operatorname{Frac}(R)$, also denote $K(R) := \operatorname{Frac}(R)$

<u>recall</u>: for any extension of fields $K \subset L$,

$$\operatorname{trdeg}_K L = \sup\{n \mid \exists\ x_1, \ldots, x_n \in L \text{ algebraically independent over } K\}.$$

e.g. $A = k[y_1, \dots, y_n]$, then $K(A) = k(y_1, \dots, y_n)$

$$\operatorname{trdeg}_k K(A) = n$$

or $A(Y) = k[x,y] / (y - x^2)$, if $Y =$ parabola $= Z(y - x^2)$

$\cong k[x]$ $\qquad K(A(Y)) = \operatorname{Frac}\left(k[x,y]/(y-x^2)\right)$

$$\cong k(x)$$

$$\operatorname{trdeg}_k K(A(Y)) = 1.$$

Proof of the theorem: First note that, by the lemma: Y irreducible $\iff I(Y)$ prime

$\iff A(Y) = A/I(Y)$ is an integral domain

Proof of $\dim Y = \dim A(Y)$ for Y affine irreducible:

The data of a chain of irreducible closed subsets of Y: $Y \supseteq Y_0 \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_n \neq \emptyset$

is equivalent (by Nullstellensatz) to the data of the chain of prime ideals

$$I(Y) \subseteq I(Y_0) \subsetneq I(Y_1) \subsetneq \cdots \subsetneq I(Y_n) \subsetneq A = k[x_1, \ldots, x_n]$$

$$\Leftrightarrow \quad (0) \subseteq \frac{I(Y_0)}{I(Y)} \subsetneq \frac{I(Y_1)}{I(Y)} \subsetneq \cdots \subsetneq \frac{I(Y_n)}{I(Y)} \subsetneq \frac{A}{I(Y)} = A(Y)$$

$$\Rightarrow \quad \dim Y = \dim A(Y)$$

Proof of $\dim Y = \dim S(Y) - 1$ for Y irreducible projective

The data of a chain of irreducible closed subsets of Y:

$$Y \supseteq Y_0 \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_n \neq \emptyset$$

is equivalent to the data of their homogeneous prime ideals (by homogeneous Nullstellensatz):

$$I(Y) \subseteq I(Y_0) \subsetneq I(Y_1) \subsetneq \cdots \subsetneq I(Y_n) \subsetneq S_+ \subsetneq S$$

$$\Longleftrightarrow \quad (0) \subseteq \frac{I(Y_0)}{I(Y)} \subsetneq \frac{I(Y_1)}{I(Y)} \subsetneq \cdots \subsetneq \frac{I(Y_n)}{I(Y)} \subsetneq S(Y)_+ \subsetneq S(Y)$$

$$\Longrightarrow \quad \dim Y = \operatorname{height} S(Y)_+ - 1$$

$$= \dim S(Y) - \dim S(Y)/S(Y)_+ - 1$$

$$S(Y) / S(Y)_+ = {}^{S}/_{I(Y)} \Big/ {}^{S_+}/_{I(Y)} \cong S/S_+ \cong k$$

$\Rightarrow \dim Y = \dim S(Y) - \text{Krull dim } k = \dim S(Y)$

Krull dim$(k) = 0$ because k has exactly one prime ideal: $(0) \subset k$.

$\square$

$$A(k) = A(\mathbb{A}^1) = k[y_1]$$

$$1 = \dim \mathbb{A}^1 = \text{Krull dim } k[y_1]$$