Def: For an ideal $I \subset k[x_0, \dots, x_n] =: S$

we denote $I_d := I \cap S_d \subset I$

$$\Rightarrow \bigoplus_{d \geq 0} I_d \subset I$$

Lemma: I is homogeneous iff $I = \bigoplus_{d \geq 0} I_d$.

Proof: exercise

Link between affine and projective:

$$\mathbb{A}^n_k = k^n \qquad \mathbb{P}^n_k = k^{n+1} \setminus \{0\} / k^*$$

$\forall\, i \in \{0, \dots, n\}$, we have injective maps

$$\varphi_i : \mathbb{A}^n \hookrightarrow \mathbb{P}^n_k$$

$$(y_1, \dots, y_n) \longmapsto (y_1, \dots, \underset{\substack{\uparrow \\ i\text{-th coordinate}}}{1}, \dots, y_n)$$

proof of injectivity: exercise.

We denote the image of φ_i by U_i:

this is the locus where $x_i = 1 \iff x_i \neq 0$ (on $\mathbb{P}^n$)

$$U_i = \{(a_0, \dots, a_n) \in \mathbb{P}^n \mid a_i \neq 0\}$$

because $(a_0, \dots, a_i, \dots, a_n) \sim \left(\frac{a_0}{a_i}, \dots, 1, \dots, \frac{a_n}{a_i}\right)$

if $a_i \neq 0$.

Note: $\mathbb{P}^n = \bigcup_{i=0}^{n} U_i$ $\left(\mathbb{P}^n = k^{n+1} \setminus \{0\} / k^*\right)$

$\forall\ (a_0, \dots, a_n), \exists i$ s.t. $a_i \neq 0$

$\Rightarrow (a_0, \dots, a_n) \in U_i$.

Also if $Y_i := \mathbb{P}^n \setminus U_i$

then $Y_i = \{(a_0, \dots, a_n) \mid a_i = 0\} = Z(x_i)$

is closed $\Rightarrow U_i$ is open.

Def: The Y_i are the coordinate hyperplanes..

In $\mathbb{A}^n$ we also have have coordinate hyperplanes:

if $k[y_1, \dots, y_n]$ is the coordinate ring of $\mathbb{A}^n = k^n$;

then $Z_i := Z(y_i)$ is a coordinate hyperplane.

Def: More generally, given a polynomial of degree 1 in $y_1, \dots, y_n$, its zero locus is called a hyperplane in k^n.

If $L \in S_1$ (i.e., L is homogeneous of degree 1)

$L \neq 0$, then $Z(L)$ is a hyperplane in $\mathbb{P}^n$.

Def: In $\mathbb{A}^n$, a hypersurface of degree d is the zero locus of a polynomial of degree d.

In $\mathbb{P}^n$, a hypersurface of degree d is the zero locus of a homogeneous polynomial of degree d.

Homogenization and dehomogenization:

e.g.: $\mathbb{P}^2$: coordinates (X, Y, Z)

$\mathbb{A}^2$: " (x, y)

$$\varphi_2 : \mathbb{A}^2 \hookrightarrow \mathbb{P}^2$$

$$(a, b) \longmapsto (a, b, 1) \qquad \text{image is } U_2 = \{(A, B, C) \mid C \neq 0\}$$

$$\varphi_2^{-1}(A, B, C) = \left(\frac{A}{C}, \frac{B}{C}\right) \longleftarrow (A, B, C) \quad C \neq 0$$

$$\left(x = \frac{X}{Z}, \ y = \frac{Y}{Z}\right)$$

Given a polynomial $P(x, y)$, we can homogenize it to obtain a homogeneous polynomial in X, Y, Z.

e.g.: $xy - 1 \rightsquigarrow \frac{X}{Z} \cdot \frac{Y}{Z} - 1$ and clear denominators $\to XY - Z^2$

This was homogenization: denote it by $H : k[x,y] \to k[X,Y,Z]$

Dehomogenization: given a homogeneous polynomial in X, Y, Z, we "evaluate it" at $(x, y, 1)$ to obtain its dehomogenization: $XY - Z^2 \rightsquigarrow xy - 1$,

denote this by $D : k[X,Y,Z] \longrightarrow k[x,y]$

We have $D \circ H = Id_{k(x,y)}$

However, $H \circ D \neq Id_{k[X,Y,Z]}$, e.g. $XYZ - Z^3$

then $H \circ D(XYZ - Z^3) = XY - Z^2$.

We have $\mathcal{Z}(H(P(x,y))) \cap U_2 = \mathcal{Z}(P(x,y)) \subset U_2$.
(exercise)

<u>Def</u>: A quasi-affine variety is an open subset of an affine variety.

<u>Def</u>: A quasi-projective variety is an open subset of a projective variety.

$\boxed{\text{Now assume } k \text{ is algebraically closed.}}$

<u>Nullstellensatz</u>: There is a 1-to-1 inclusion reversing correspondence between affine varieties (or algebraic subsets of $\mathbb{A}^n$) and radical ideals in

$A := k[y_1, \ldots, y_n]$; given by

$$Y \longmapsto I(Y)$$

$$Z(I) \longleftarrow I$$

Recall: An ideal is radical if $I = \sqrt{I}$, where

$$\sqrt{I} := \{ P \in A \mid \exists\, m \in \mathbb{N} \text{ s.t. } P^m \in I \} \supset I.$$

Homogeneous nullstellensatz: There is a 1-to-1 inclusion reversing correspondence between projective varieties (or algebraic subsets of $\mathbb{P}^n$) and homogeneous radical ideals in $S_+ := \bigoplus_{d>0} S_d$ given by

$$Y \longmapsto I(Y)$$

$$Z(I) \longleftarrow I$$

where $Z(I)$ is the set of common zeros of all the homogeneous elements of I.

Dimension: $\dim \mathbb{A}^n = \dim k^n = n$

$\dim \mathbb{P}^n = \dim k^{n+1} - 1 = n$

We want to define the dimension of an arbitrary variety. (= affine or quasi-affine or projective or quasi-projective)

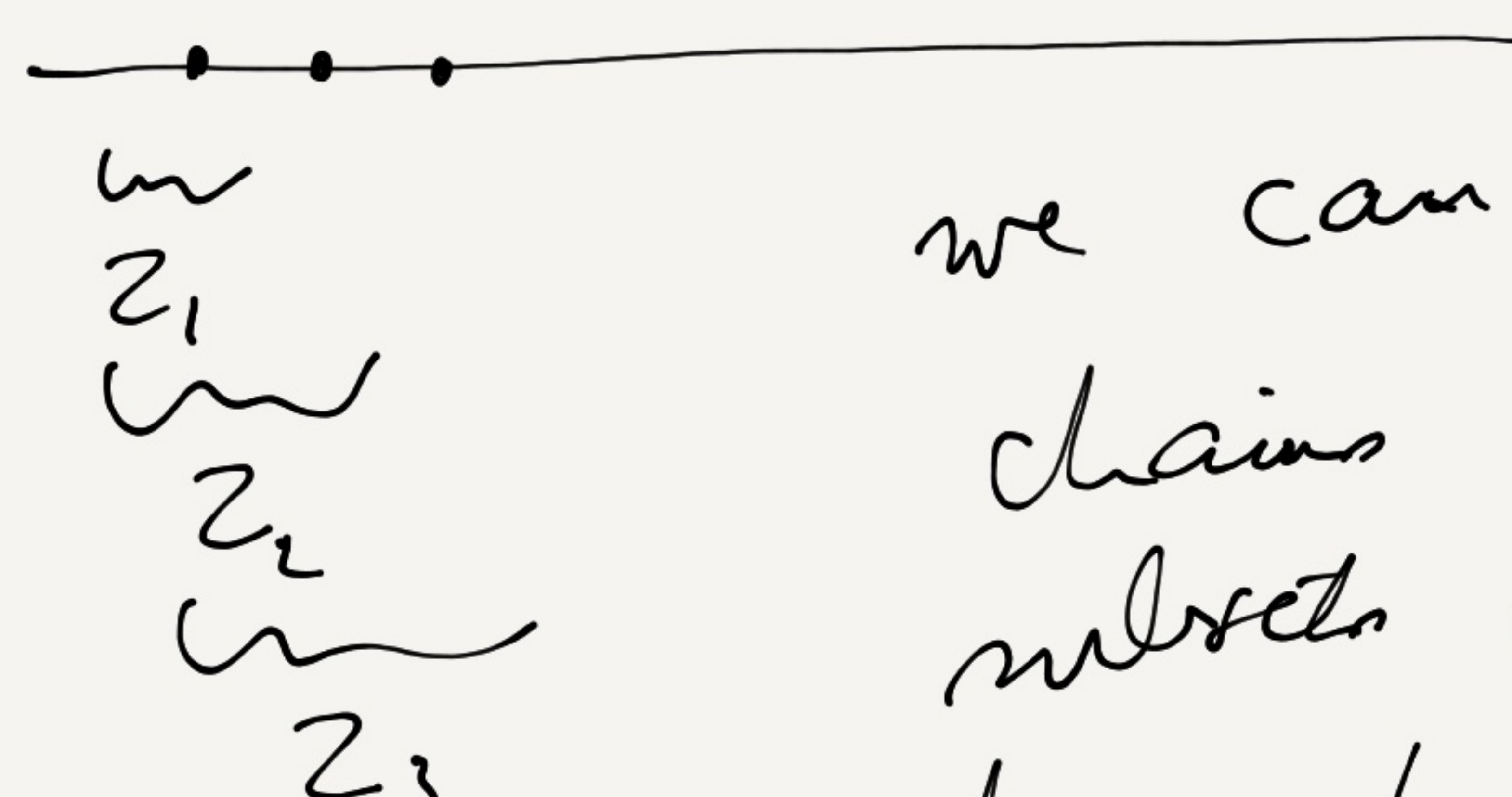
in k^n, if $Y := Z(y_0 + y_1 + y_2 - y_n) \subset k^n$ should have dimension $n-1$

$Y \supset Y \cap Z(y_0)$ should have dim. $n-2$.

essentially, we could do this n times.

This gives us the idea of defining the dimension of a variety in terms of descending chains of closed subsets.

For this to make sense, we need the idea of irreducibility:



we can get descending chains of closed subsets of any length if we don't require irreducibility $\Rightarrow \dim k \geq n$ $\forall n$.

<u>Def</u>: A topological space Y is called irreducible if $Y \neq \emptyset$ and $\forall$ Y_1, Y_2 non-empty closed subsets of Y, we have

$$Y_1 \cup Y_2 = Y \implies Y_1 = Y \text{ or } Y_2 = Y$$

<u>Definition:</u> A topological space is called noetherian if it satisfies the descending chain condition for closed subsets, i.e., $\forall$ sequence of closed subsets

$$Y_1 \supseteq Y_2 \supseteq \dots \supseteq Y_n \supseteq \dots$$

$\exists$ n s.t. $\forall$ $r \geq n$ $Y_r = Y_n$.