TEXed on 1/21/1999

Almost All Rooted Maps Have Large Representativity

Edward A. Bender
Center for Communications Research
4320 Westerra Drive
San Diego, CA 92121, USA

Zhicheng Gao* and L. Bruce Richmond*
Department of Combinatorics and Optimization
University of Waterloo
Waterloo, Ontario
Canada N2L3G1

* Research supported by the NSERC under Grant A-4067

AMS-MOS subject classifications: 05C10, 05C30, 05C15

ABSTRACT

Let M be a map on a surface S. The edge-width of M is the length of a shortest non-contractible cycle of M. The face-width (or, representativity) of M is the smallest number of intersections a noncontractible curve in S has with M. (The edge-width and face-width of a planar map may be defined to be infinity.) A map is an LEW-embedding if its maximum face valency is less than its edge-width. For several families of rooted maps on a given surface, we prove that there are positive constants c_1 and c_2, depending on the family and the surface, such that

1. almost all maps with n edges have face-width and edge-width greater than $c_1 \log n$ and

2. the fraction of such maps which are LEW-embeddings and the fraction which are not LEW-embeddings both exceed n^{-c_2}.

1. Introduction

We begin with some definitions:

- A *map* is a connected graph G embedded in a surface S (a closed 2-manifold) such that all components of $S - G$ are simply connected regions. These components are called *faces* of the map. As is usual with maps, loops and multiple edges are permitted. A loopless map is a map without loops. A simple map is a map without loops and multiple edges.

- A map is *rooted* if an edge is distinguished together with a vertex on the edge and a side of the edge. *Unless stated otherwise, all maps shall be rooted.*

- We use Tutte's definition [16] of connectivity, a graph or corresponding map is k-connected (abbreviated k-c) if the girth is at least k and it requires removing at least k vertices to separate the graph. It follows that a 2-c map has no loops and a 3-c map has no multiple edges.

- A *facial walk* in a map is a closed walk along the boundary of a face. The length of the facial walk is called the valency of the face. By a *cycle* in a map, we mean a simple closed curve consisting of edges of the map.

- A cycle is called *contractible* if it is homotopic to a point, otherwise it is called *non-contractible* and denoted by *nc-cycle*. The *edge-width* of a map M, denoted by $\mathrm{ew}(M)$, is the length of a shortest nc-cycle in M. The *face-width* (or *representativity*) of a map M, denoted by $\mathrm{fw}(M)$, is the smallest number of intersections a noncontractible curve in the surface has with the map. A *large-edge-width* embedding (abbreviated LEW-embedding) is a map whose edge-width exceeds its maximum face valency.

- An edge is called *singular* if it incident with only one face. A face is called *singular* if its facial walk is not a cycle. A map is called *non-singular* if it has no singular faces. A map is called *non-bisingular* if there are no faces f_1, f_2 and vertices v_1, v_2 such that f_i and v_j are incident for all choices of i and j, except we allow the four incidences to occur around one edge (at both ends).

- Let $\mathcal{F}$ be some family of rooted maps and let $\mathcal{F}_n(S)$ denote the set of n-edged maps in $\mathcal{F}$ that lie on a surface S. We will say that $\mathcal{F}$ *grows normally* if

$$|\mathcal{F}_n(S)| \sim A n^{-5\chi/4} \rho^n$$

where χ is the Euler characteristic of S, the limit is taken through those n for which $\mathcal{F}_n(S) \neq \emptyset$, $A = A(S, \mathcal{F})$ depends only on S and $\mathcal{F}$, and $\rho = \rho(\mathcal{F})$ depends only on $\mathcal{F}$.

Robertson and Vitray [12] have shown that embeddings with large face-width share many properties with planar embeddings. For example, if a 2-c graph G has an embedding π in an orientable surface of genus g with $\mathrm{fw}(G) \geq 2g + 3$, then any other embedding of G in the same surface is obtained from π by a sequence of "2-switchings" (defined by Whitney [18] who proved the planar case). It follows from this that if a 3-c graph G has an embedding with large face-width in an orientable surface, then it has a unique embedding in that surface and every automorphism of G extends to a map automorphism of that embedding. Thomassen [13] has shown that LEW-embeddings also share many properties with planar embeddings. Recently, Thomassen has also shown that if a map has large edge-width then it is vertex 5-colorable.

Various authors have shown that a variety of families of rooted maps grow normally. We study the edge-widths and face-widths of maps in a normally growing family, showing roughly that the edge-width and the face-width of most such maps are at least the logarithm of the number of edges. For convenience, we collect here a list of known families of normally growing rooted maps.

Proposition 1. *Let D be a finite subset of the positive integers. The normal growth of the following families of rooted maps is proved in the locations cited.*

1. *rooted maps [1];*
2. *rooted smooth maps [1];*
3. *rooted loopless maps, rooted simple maps and rooted 3-c triangulations [10];*
4. *rooted nonsingular maps and rooted 2-c maps (and the numbers are asymptotically equal) [5];*

5. *rooted triangular maps [7];*

6. *rooted triangulations, that is, rooted 2-c triangular maps [8];*

7. *for certain D as discussed in [9], rooted maps with all face valencies in D ($D = \{3\}$ is case 4).*

We expect that 3-c maps will be added to this list in the near future. The following proposition adds various families of quadrangulations to the list.

Proposition 2. *On any given surface S, there is a bijection ϕ between n-edged maps and n-faced bipartite quadrangulations, such that* $\mathrm{fw}(M) = \mathrm{ew}(\phi(M))/2$.

Proof: The proof is a straightforward extension of the bijection on the sphere given by Brown [6]: For any map M, place a vertex in each face and join it to the vertices on the boundary of the face through every corner and remove all the original edges of M. This gives a bipartite quadrangulation Q, whose root corner can be chosen the same as the root corner of M. This is clearly a bijection, and any nc-cycle of length $2k$ in Q intersects $G(M)$ in exactly k vertices. ∎

We are grateful to Nick Wormald for his helpful observations.

2. Statement of results

We will prove the following:

Theorem 1. *Let $\mathcal{F}$ be one of the families in Proposition 1. For every $\epsilon > 0$, there is a $\delta = \delta(\epsilon, S) > 0$ such that the fraction of maps M in $\mathcal{F}_n(S)$ that have* $\mathrm{ew}(M) < \delta \log n$ *is $o(n^{-1/4+\epsilon})$ as $n \to \infty$. In particular, almost all maps M in $\mathcal{F}_n(S)$ have* $\mathrm{ew}(M) \geq \delta(1/4, S) \log n$ *as $n \to \infty$.*

Corollary 1. *Let $\mathcal{F}$ be one of the families in Proposition 1. For every $\epsilon > 0$, there is a $\delta = \delta(\epsilon, S) > 0$ such that the fraction of maps M in $\mathcal{F}_n(S)$ that have* $\mathrm{fw}(M) < \delta \log n$ *is $o(n^{-1/4+\epsilon})$ as $n \to \infty$. In particular, almost all M maps in $\mathcal{F}_n(S)$ have* $\mathrm{fw}(M) \geq \delta(1/4, S) \log n$ *as $n \to \infty$.*

Corollary 2. *Fix $\epsilon > 0$. For any family of maps in Proposition 1 that are required to have bounded face valencies, the fraction of maps in $\mathcal{F}_n(S)$ which are not LEW-embeddings is bounded by $o(n^{-1/4+\epsilon})$ as $n \to \infty$.*

We conjecture that "5-colorable" could be replaced by "4-colorable" in the following corollary. This would be best possible since almost all maps contain a submap requiring 4 colors [3, 4].

Corollary 3. *On any given surface:*

1. *almost all loopless maps are vertex 5-colorable, and dually, almost all maps without singular edges are face 5-colorable;*
2. *almost all simple maps are vertex 5-colorable;*
3. *almost all 2-c maps are vertex 5-colorable and face 5-colorable.*

Using Corollary 3 and the standard construction of a k-flow from a face k-coloring (as in [11]), we have established Tutte's 5-flow conjecture [17] for a large class of graphs that do not have bridges.

In contrast to Corollary 1, we have only been able to obtain weak results on the number of LEW-embeddings when face valencies are not bounded:

Theorem 2. *Let $\mathcal{F}$ be either 1-c, smooth or 2-c rooted maps and let S be a surface other than the sphere. There is a $c = c(S)$ such that, for all sufficiently large n, the fraction of maps in $\mathcal{F}_n(S)$ which are LEW-embeddings is at least n^{-c} and the fraction which are not LEW-embeddings is also at least n^{-c}. For almost all maps in $\mathcal{F}_n(S)$, the ratio of the maximum face valency to the edge-width is bounded above by a constant independent of n.*

Suppose we are dealing with a class of maps which allows unbounded face valency. In view of face size estimates as in [2], it seems reasonable to expect that the probability distribution for face valencies is roughly geometric. Thus, there should be a constant $c > 0$ so that for all $\epsilon > 0$

$$\text{Prob}\left\{\left|\frac{\text{maximum face valency}}{\log n} - c\right| > \epsilon\right\} = o(1) \quad \text{as } n \to \infty, \tag{1}$$

where the probability is based on a uniform distribution over $\mathcal{F}_n(S)$. The proof of Theorem 1 shows that noncontractible cycles can be identified with faces of certain maps. Thus, it seems reasonable that a similar result to (1) should hold for the edge-width, though probably with some other constant c'. If $c' > c$, then almost all maps in the class will be LEW-embeddings and, if $c' < c$, then almost no maps in the class will be. We suspect that $c' < c$. Unfortunately, we cannot prove any of these speculations.

3. Proofs

Lemma 1. *Let $a_n \geq 0$ and $b_n \geq 0$ satisfy*

$$a_n = O(n^\alpha R^n) \quad \text{and} \quad b_n = O(n^\beta R^n)$$

for some constants α, β and $R > 1$ and let

$$c_n = \sum_{n_1+n_2=n} a_{n_1} b_{n_2}.$$

We have

1. *if $\alpha > -1$ and $\beta > -1$, then $c_n = O(n^{\alpha+\beta+1} R^n)$;*
2. *if $\alpha < -1$ and $\beta \geq \alpha$, then $c_n = O(n^\beta R^n)$.*

Proof: For any power series $A(x) = \sum a_n x^n$ and $B(x) = \sum b_n x^n$, we use $[x^n]A(x)$ to denote a_n and $A(x) = O(B(x))$ to denote $a_n = O(b_n)$. Assume that $\alpha > -1$ and $\beta > -1$. Using [15, p. 242, Ex. 8], we have

$$[x^n](1 - Rx)^{-(\delta+1)} \sim \frac{n^\delta R^n}{,\,(\delta + 1)} = O\left(n^\delta R^n\right) \quad \text{for} \quad \delta > -1.$$

Therefore $A(x) = O\left((1 - Rx)^{-(\alpha+1)}\right)$ and $B(x) = O\left((1 - Rx)^{-(\beta+1)}\right)$. Since $A(x)$ and $B(x)$ have non-negative coefficients, we have

$$A(x)B(x) = O\left((1 - Rx)^{-(\alpha+\beta+2)}\right).$$

Using [15, p. 242 Ex. 8] again, we obtain the first part of the lemma.

We have

$$\begin{aligned} c_n &= \sum_{n_1+n_2=n} a_{n_1} b_{n_2} = O\Big(R^n \sum_{n_1+n_2=n} n_1^\alpha n_2^\beta \Big) \\ &= O\Big(R^n \sum_{n_1=1}^{[n/2]} n_1^\alpha (n-n_1)^\beta \Big) + O\Big(R^n \sum_{n_1=[n/2]+1}^{n} n_1^\alpha (n-n_1)^\beta \Big). \end{aligned}$$

For $\alpha < -1$ and $\beta \geq \alpha$, we have

$$\sum_{n_1=1}^{[n/2]} n_1^\alpha (n-n_1)^\beta = O\left(n^\beta\right),$$

$$n_1^\alpha (n-n_1)^\beta \leq n_1^\beta (n-n_1)^\alpha \quad \text{for} \quad n_1 \geq [n/2]+1,$$

and thereby,

$$\sum_{n_1=[n/2]+1}^{n-1} n_1^\alpha (n-n_1)^\beta = O\Big(\sum_{n_1=[n/2]+1}^{n-1} n_1^\beta (n-n_1)^\alpha \Big) = O\left(n^\beta\right).$$

Therefore, $c_n = O\left(n^\beta R^n\right)$ in this case ∎

Proof of Theorem 1 for non-singular maps: We use the following standard topological fact: If C is a cycle in a map of Euler characteristic χ, then cutting the surface along C, duplicating each edge of C and sewing in disks for the created holes gives one of the following three possibilities. (We can imagine edges having width and then cut through the middle of each edge of C.)

- If C is contractible, then we have one planar map and another map of characteristic χ.
- If C is an nc-cycle and the result is not connected, then we have two non-planar maps with the sum of their characteristics being $\chi + 2$.
- If C is an nc-cycle and the result is connected, then we have one map with type $\chi + 1$ or $\chi + 2$, depending on whether the cut destroys a cross cap or a handle.

For any map M containing an nc-cycle C of length l, apply the cutting process just described. At least one of the following three possibilities occurs: (1) the result is not connected, (2) a cross cap destroyed and (3) a handle destroyed. We consider each and add the results, thereby obtaining an overcount of maps with given edge-widths.

1. *Not Connected.* The process yields two non-planar rooted maps M_1 and M_2, where M_1 has a second root edge bordering some face with the same valency l as M_1's root face. It is clear that the cutting process does not create singular faces, hence M_1 and M_2 are all non-singular. We can reverse this process by identifying the second rooted face of M_1 and the root face of M_2, root edge to root edge, so that the two faces of valency l disappear. For a rooted map with n edges, there are $4n$ ways to choose a second root edge. Using normal growth, we see that the number of maps obtained in this way is bounded by

$$\sum_{j=\chi+1}^{1} \sum_{n_1+n_2=n+l} O(n_1 n_1^{-5j/4} \rho^{n_1}) O(n_2^{-5(\chi-j+2)/4} \rho^{n_2}) \\ = O\left((n+l)^{2-5(\chi+2)/4} \rho^{n+l}\right) + O\left((n+l)^{1-5(\chi+1)/4} \rho^{n+l}\right), \tag{2}$$

where the first term on the right hand side comes from Lemma 1(1) applied to the terms with $j > \chi + 1$ and the second term comes from Lemma 1(2) applied to the terms with $j = \chi + 1$. By normal growth and $l = O(n)$, this sum is

$$O\left(\rho^l n^{-1/4} |\mathcal{F}_n(S)|\right). \tag{3}$$

2. *Cross Cap Destroyed.* The process yields a rooted map of characteristic $\chi + 1$ with a distinguished face of valency $2l$. It is clear that the cutting process does not create singular faces, hence the resulting map is also non-singular. We can reverse the process by identifying the opposite edges on the boundary of the distinguished face in such a way that the face disappears. There are $O(n + l)$ ways to choose a distinguished face. Using normal growth, we see that the number of maps obtained in this way is bounded by

$$O\left((n+l)(n+l)^{-5(\chi+1)/4} \rho^{n+l}\right) = O\left(\rho^l n^{-1/4} |\mathcal{F}_n(S)|\right). \tag{4}$$

3. *Handle Destroyed.* The process yields a rooted map of characteristic $\chi + 2$ with two extra rooted faces of valency l. Since the cutting process does not create singular faces,

the resulting map is also non-singular. We can reverse the process by identifying the boundaries of two extra rooted faces, root edge to root edge, in such a way that the faces disappear. Using normal growth, we see that the number of maps obtained in this way is bounded by

$$O\left((n+l)^2(n+l)^{-5(\chi+2)/4}\rho^{n+l}\right) = O\left(\rho^l n^{-1/2}|\mathcal{F}_n(S)|\right).$$

Since the sum of ρ^l over $l < \delta \log n$ is $O(\rho^{\delta \log n}) = O(n^{\delta \log \rho})$, we complete the proof for nonsingular maps by choosing $\delta < \epsilon / \log \rho$. ∎

Proof of rest of Theorem 1: The case of 2-c maps follows from the asymptotic equality of nonsingular and 2-c maps [5]. The cases of 1-c maps is simpler since we do not need to justify the 2-connectivity in the various cases. For triangular maps the cutting process yields near-triangular maps with one or two distinguished faces of valency l. Adding a vertex to the interior of each distinguished face and joining the vertex to all vertices on the boundary of the distinguished face creates a triangular map with $2l$ more edges and one distinguished vertex for each distinguished face. The cutting process can not create loops and thereby destroy 2-connectivity. There are $O(n)$ ways to choose a distinguished vertex so the proof above is easily modified to handle the cases of 1-c or 2-c triangular maps. The case of general valency restrictions can be handled similarly. ∎

Remark 1. If S is orientable, we may replace $1/4$ by $1/2$ in Theorem 1 and its corollaries because the sum in (2) ranges over $\chi + 2 \le j \le 0$ and (4) never occurs.

Remark 2. If 3-c maps are shown to be in normal families, the above proof is easily modified to include them: When we sew in the disks we may create bisingularities; however, adding a vertex to the interior of each distinguished face, then joining the vertex to all vertices on the boundary of the distinguished face creates a map with an extra $2l$ edges and no bisingularities. This does not significantly affect the proof.

Proof of the corollaries: Corollary 1 follows immediately from Proposition 2 and Theorem 1. Corollary 2 is immediate. Thomassen [14, Thm. 5.7] has shown that, on a surface of genus g, maps with edge-width at least 2^{4g+8} are 5-colorable. Corollary 3 follows at once from Theorem 1, Thomassen's result and the fact that the edge-width is always no less than the face-width and the face-width is invariant under the dual operation. ∎

Lemma 2. *Let $\mathcal{F}$ be a family of rooted maps closed under the operation of adding a vertex to the interior of a face and then joining the vertex to at least three vertices on the boundary of the face in cyclic order. Assume that $|\mathcal{F}_{n+1}(S)| \le A|\mathcal{F}_n(S)|$ for some A and all sufficiently large n. Then there are positive constants δ, B_1 and B_2, depending on A but not on S, such that*

1. *the fraction of maps in $\mathcal{F}_n(S)$ which have root face valency t is at most $B_1 e^{-\delta t}$ for all sufficiently large n.*
2. *the fraction of maps in $\mathcal{F}_n(S)$ which have maximum face valency greater than t is at most $B_2 n e^{-\delta t}$ for all sufficiently large n.*

Proof: The proof of the first conclusion is an easy modification of that of [5, Lemma 4] and is included for completeness. We used face operations instead of adding face diagonals as in [5] because adding face diagonals may create multiple edges and thereby destroy 3-connectivity. We just prove the lemma for 3-c maps. The proofs for 1-c and 2-c maps are similar.

Given a 3-c rooted map, add a vertex to the interior of the root face, join it to the root vertex and any other $k \ge 2$ vertices on the boundary of the root face in cyclic order. Keep the original rooting. This process is clearly reversible. Thus we obtain $|\mathcal{F}_{n,t}(S)|\binom{t-1}{k}$ different 3-c rooted maps with $n+k$ edges, where $\mathcal{F}_{n,t}(S)$ is the subset of maps in $\mathcal{F}_n(S)$ with root face valency t. Thus

$$|\mathcal{F}_{n,t}(S)|\binom{t-1}{k} \le |\mathcal{F}_{n+k}(S)| \le A^k|\mathcal{F}_n(S)|$$

and so, since $\binom{m}{k} > (m/k)^k$ for $0 < k < m$,

$$\frac{|\mathcal{F}_{n,t}(S)|}{|\mathcal{F}_n(S)|} \le \frac{A^k}{\binom{t-1}{k}} \le \left(\frac{Ak}{t-1}\right)^k.$$

With $k = [(t-1)/(A+1)]$ and $t \to \infty$, the first part of the lemma follows immediately.

Look at those maps in $\mathcal{F}_n(S)$ for which the maximum face valency occurs at the root face. Since there are at most $4n$ ways to reroot such a map, the rest of the lemma follows from the first part. ∎

Proof of Theorem 2: By normal growth and Lemma 2(2), there are

$$n^{-5/2}\rho^n(1+o(1)) \tag{5}$$

maps in $\mathcal{F}_n$(sphere) which have maximum face valency less than $y \log n$ when y is sufficiently large. Fix such a y. Call this set of maps $\mathcal{F}_n^*$.

We now show that there is an LEW-embedding $M_1(n)$ on S with $O(\log n)$ edges and edge-width at least $y \log n$. Start with an LEW-embedding on S with k edges and edge-width l. (By Theorem 1 for triangulations, LEW-embeddings always exist.) Insert $m-1$ vertices on every edge. The resulting map is still an LEW-embedding which has km edges and edge-width lm. Let m be $y \log n/l$ rounded up and let $e(y,n)$ be the number of edges in the resulting map. Clearly $e(y,n) = O(\log n)$.

We can join $M_1(n)$ and any map M together by replacing the root edge of M_1 with a digon, placing M inside the digon and identifying its root edge with the root edge of the altered $M_1(n)$. The face valencies of the resulting map are the face valencies of M and $M_1(n)$, and the edge-width equals the edge-width of $M_1(n)$. Do this for all $M \in \mathcal{F}_{n-e(y,n)}^*$. Since $M_1(n)$ is fixed the process is reversible and so we obtain $|F_{n-e(y,n)}^*|$ maps in $\mathcal{F}_n(S)$ which are LEW-embeddings. Since $e(y,n) = O(\log n)$, we are done by (5) and normal growth.

We now construct a fraction n^{-c_2} of maps with n edges on S which are not LEW-embeddings. Let $M_2 \in \mathcal{F}(S)$ not be an LEW-embedding. (An M_2 can be constructed by inserting a big cycle in a face of a map.) Let M_2 have k edges. We can attach M_2 and an $M \in \mathcal{F}_{n-k}$(sphere) as described in the previous paragraph to obtain maps in $\mathcal{F}_n(S)$ which are not LEW-embeddings. We are done by normal growth.

The second part of Theorem 2 follows immediately from Theorem 1 and Lemma 2(2). ∎

4. References

[1] E.A. Bender and E.R. Canfield, The asymptotic number of rooted maps on a surface. *J. Combin. Theory, Ser. A*, **43** (1986) 244–257.

[2] E.A. Bender and E.R. Canfield, Face sizes of 3-polytopes. *J. Combin. Theory, Ser. B*, **46** (1989) 58–65.

[3] E.A. Bender, Z. Gao and L.B. Richmond, Submaps of maps I: General 0-1 laws. *J. Combin. Theory, Ser. B*, **55** (1992) 104–117.

[4] E.A. Bender and L.B. Richmond, Submaps of maps III: k-connected nonplanar maps. *J. Combin. Theory, Ser. B*, **55** (1992) 125–132.

[5] E.A. Bender and N.C. Wormald, The asymptotic number of rooted nonseparable maps on a surface. *J. Combin. Theory, Ser. A*, **49** (1988) 370–380.

[6] W.G. Brown, Enumeration of quadrangular dissections of the disk. *Canad. J. Math.* **17** (1965) 302–317.

[7] Z. Gao, The number of rooted triangular maps on a surface. *J. Combin. Theory, Ser. B*, **52** (1991) 236–249.

[8] Z. Gao, The asymptotic number of rooted 2-connected triangular maps on a surface. *J. Combin. Theory, Ser. B*, **54** (1992) 102–112.

[9] Z. Gao, The number of degree restricted maps on general surfaces, preprint.

[10] Z. Gao, A pattern for the asymptotic number of rooted maps on surfaces, *J. Combin. Theory, Ser. A*, to appear.

[11] F. Jaeger, Nowhere-zero flow problems. In L.W. Beineke and R.J. Wilson (eds.) *Selected Topics in Graph Theory* **3** Academic Press, London (1988) 71–95.

[12] N. Robertson and R. Vitray, Representativity of surface embeddings, *Algorithms and Combinatorics* **9** (1990) 293–328.

[13] C. Thomassen, Embeddings of graphs with no short non-contractible cycles. *J. Combin. Theory, Ser. B*, **48** (1990) 155–177.

[14] C. Thomassen, 5-coloring maps on surfaces. *J. Combin. Theory, Ser. B*, to appear.

[15] E.L. Titchmarsh, *The Theory of Functions*, 2nd ed., Oxford Univ. Press, London (1939).

[16] W.T. Tutte, A theory of 3-connected graphs. *Konink. Nederl. Akad. Van W. Proc.* **64** (1961) 441-455.

[17] W.T. Tutte, A contribution to the theory of chromatic polynomials. *Canad. J. Math.* **6** (1954) 80–91.

[18] H. Whitney, Congruent graphs and the connectivity of graphs. *Amer. J. Math.* **54** (1932) 150–168.