## Some Unsolved Problems in Map Enumeration

Edward A. Bender

**Abstract.** This is a brief introduction to several problems related to the enumeration of maps in surfaces. The problems range from requests for bijective proofs to asymptotic problems to an algebraic question.

## 1. A Very Brief Introduction to Map Enumeration

For many years, people have considered various problems in the enumeration of graphs, beginning with the enumeration of trees. Extensive work on the enumeration of maps did not begin until Tutte's work in the 1960's; however, one map enumeration problem—the determination of the number of combinatorially inequivalent 3-dimensional convex polytopes—is over a century old.

Traditionally an *map* has been thought of as a connected unlabeled graph which has been embedded in (= continuous injection into) a sphere. In the 1960's various classes of (rooted) maps were enumerated thanks to the pioneering work of Tutte. His approach consists of three main steps:

1. Enumerate *rooted maps* instead of those originally requested. A map is rooted if an edge, a direction on the edge, and a side of the edge are distinguished. Rooting a map destroys symmetries so that we are essentially faced with the easier problem of counting labeled objects. (Mathematicians pretty much abandoned as too difficult the original problem of counting unrooted maps, but some work has been done on it.)
2. The side of the root edge distinguishes a face, the *root face* of the map. The *degree* of a face is the number of edges (with multiplicity) that border on the face. If $a(n,k)$ is the number of maps in some class with $n$ edges and root face degree $k$, then it is possible to obtain an equation for $A(x,y) = \sum a(n,k)x^n y^k$ by means of a recursive construction for maps. This recursive construction is usually found by looking at what happens when the root edge is removed.
3. The above method usually leads to a quadratic equation for $A(x,y)$ that contains $x$, $y$, and $A(x,1)$. The equation must have a unique power series solution because it was derived from a recursive construction.

   The third step is solving this equation. Setting $x = 1$ gives only a trivial identity. The key is that the quadratic has a solution of the form $-B \pm Q^{1/2}$. Thus $Q = Q(x,y,A(x,1))$ must be the square of a power series. Brown formalized the technique for solving this equation in the "quadratic method" [11].

   Let $y = y(x)$ be a function such that $Q(x,y(x),A(x,1)) = 0$. (We currently do not know $y(x)$.) Since we must have $Q = R^2$ for some (unknown) power series $R$, it follows that $R(x,y(x),A(x,1)) = 0$ and so

$$Q_y(x,y,A(x,1)) = 2R(x,y(x),A(x,1))R_y(x,y(x),A(x,1)) = 0.$$

   Thus we have two equatins in the two unknown functions $y(x)$ and $A(x,1)$:

$$Q(x,y,A(x,1)) = 0 \quad\text{and}\quad Q_y(x,y,A(x,1)) = 0. \tag{1.1}$$

With some ingenuity, these equations can be manipulated to obtain a parametric representation for $A(x,1)$. Lagrange inversion then yields a (possibly unwieldy) formula for $a(n) = \sum_k a(n,k)$ and asymptotic methods (hopefully) lead to estimates for $a(n)$.

Of course, once one class of maps has been enumerated, it may be possible to enumerate another by functional composition. For example, suppose the class of all maps and the class of trees have been enumerated. *Smooth maps* are maps with no vertices of degree one. One can build up all maps in a unique manner from smooth maps by attaching trees in the "corners" of smooth maps. This leads to a functional equation for all maps in terms of smooth maps and trees. Inversion of this equation leads to a formula for smooth maps.

Maps in general surfaces were looked at only fitfully from the 1960's until the mid 1980's. (The major papers were a study of 2-connected maps on the projective plane by Brown [12] and a series of papers by Walsh and Lehman [20].) To extend the definition of a map to a compact surface $S$ without boundary we require that the graph $G$ be embedded in $S$, say $m(G) \subset S$ such that $S - m(G)$ consists of regions homeomorphic to discs, called the faces of the map $m(G)$. This implies that the graph is connected and also that the genus does not go to waste by simply inducing a handle or crosscap in a face. Some people talk of maps "on a surfac" and others of "maps in a surface."

A fairly extensive bibliography on the enumeration of maps up until 1984 can be found in [7].

## 2. Bijective Problems

The skeleton of a 3-dimensional convex polytope is a 3-connected graph which can be viewed as a map in the sphere. Conversely, every such map is associated with such a polytope. Let $p(i,j)$ be the number of such rooted polytopes having $i+1$ vertices and $j+1$ faces. Mullin and Schellenberg [16] obtained a parametric representation for the generating function ofr $p(i,j)$ using Tutte's method. (See [2] for a discussion.) Wormald and I [8] were able to obtain a remarkably simple asymptotic formula from their generating function:

$$p(i,j) \sim \frac{1}{3^5 ij}\binom{2i}{j+2}\binom{2j}{i+3} \quad \text{uniformly as } \max(i,j) \to \infty, \tag{2.1}$$

where we agree that $0 \sim 0$.

**Problem 1.** *Such a simple asymptotic formula may well have a less round about proof; hopefully even one in which bijections play a major role. Is this the case or is it just wishful thinking?* In his thesis Visentin obtained a remarkable equality between maps and quadrangulations in orientable surfaces. A quadrangulation is a map in which each face has degree 4. We will look at one aspect of this equality. Let $Q_g(n)$ ge the number of $n$-faced rooted quadrangulations in the *orientable* surface of genus $g$. Let $M_g(n,t)$ be the number of $n$-edged rooted maps in the surface of genus $g$ having a $t$-set of marked vertices. Corollary 5.2.2 of [15] specializes to

$$Q_g(n) = \sum_{i=0}^{g} 2^{2i} M_i(n, 2g-2i). \tag{2.2}$$

The proof of (2.2) is based on group characters.

**Problem 2.** *Find a bijective proof of (2.2). (One has long been known for $g = 0$; i.e., the sphere.)*

| |
|---|
| Brown and Jackson [10] give a bijection. |

It seems reasonable to expect a generalization of (2.2) to nonorientable surfaces. Richmond, Wormald, and I have looked at this problem a bit and found a bijective proof for

$$Q_{1/2}(n) = 2M_{1/2}(n, 0) + M_0(n, 1), \tag{2.3}$$

where the 1/2 refers to the projective plane. We also showed that a bijective proof of (2.2) for the torus would lead to a bijective proof of

$$Q_{\mathrm{KB}}(n) = 4M_{\mathrm{KB}}(n, 0) + 2M_{1/2}(n, 1) + 2M_0(n, 2) \tag{2.4}$$

for the Klein bottle. Since methods exist for computing generating functions for the various counts mentioned here, one could always prove (2.3) and (2.4) by generating function arguments. Also, one could look at generating functions for maps in nonorientable surfaces of higher genus to try to find a generalization of (2.3) and (2.4). That has not been done.

**Problem 3.** *Is there a reasonable generalization of (2.2) to all surfaces?* Often in map enumeration, it is better to deal with all surfaces of a given Euler characteristic and then subtract the results for an orientable surface, if any. That's the reason I used "all" instead of "nonorientable" in this problem. I suspect that the answer will be "No" and than one can obtain support for it by looking at generating functions.

Let $D$ be an arbitrary set of positive integers and let $m_D(n)$ be the number of $n$-edged rooted maps in the sphere for whcih every face degree belongs to $D$. Canfield and I [5] studied $m_D(n)$. If $\gcd(D)$ is even, our result has a particularly simple form:

$$(n+1)m_D(n) = [x^{n+2}]\ (R(x)^2), \tag{2.5}$$

where

$$2R(x) = 2x + x \sum_{i \in D} \binom{i}{i/2} (R(x))^{i/2}$$

determines the power series $R(x)$ recursively—simply plug $R(x)$ truncated at degree $k$ into the right side and then truncate the left side at degree $k+1$ to obtain one more coefficient of $R(x)$. (Here $[x^m]\, f(x)$ is the coefficient of $x^m$ in $f(x)$.) The recursive formula for $R(x)$ has a tree-like flavor.

**Problem 4.** *The formula for $m_D(n)$ suggests that there should be a bijective proof based on joining together two copies of some thing that is counted by $R(x)$. Find such a bijection.*

| |
|---|
| A nice bijective proof has been found by Schaeffer [19]. His bijection also facilitates random generation. |

## 3. Algebraic Problems

The work [5] that led to (2.5) was rather round about. First, suppose that $\max D = t < \infty$. Using a Tutte type decomposition, an equation for the degree restricted maps can be found that involves, as expected, counting them by the number of edges and root face degree. As usual, the equation is quadratic, but unfortunately it is complicated because of degree restrictions: it involves the unknowns $M_{D,k}(x) = \sum m_D(n,k)x^n$ for various root face degrees $k$. Unfortunately we have more unknowns than can be determined by the two equations supplied by the quadratic method. By studying congruences modulo $2^{2t-1}$ and showing that the simplest possible way to obtain a solution actually works, we obtained a generating function form $m_D(n)$. We then let $t \to \infty$.

**Problem 5.** *The approach in [5], while perhaps amusing technically, is unsatisfyingly ad hoc. Does there exist a general algebraic tool like the quadratic method [11] lurking in the background?* Although some intriguing asymptotic patterns have been found for maps on surfaces (see the next section), this is not the case with exact formulae. In [4] it is shown that the generating function for maps by number of edges on an orientable surface of genus $g > 0$ is

$$M_g(x) = \frac{(\rho-1)^{2g}P(\rho)}{\rho^{5g-3}(\rho+2)^b(\rho+5)^c} \quad \text{where} \quad \rho = \sqrt{1-12x},$$

$b$ and $c$ are nonnegative integers, and $P$ is a polynomial that does not vanish at 0 or 1. The values of $M_g(x)$ for $g = 1,\ 2,\ 3$ suggest additional properties of $M_g(x)$, but $P(\rho)$ appears to be quite messy.

> **Problem 6.** *Perhaps there is some simple formula for $M_g(x)$. Try to find and prove such a formula or convince your colleagues that one probably does not exist.*

| |
|---|
| There may well be no simple formula. Arquès and Giorgetti [1] may have done as much as possible. |

## 4. Asymptotic Problems

A variety of classes of maps have been studied asymptotically in surface of arbitrary genus. (See [14].) Of course, one can always ask for the asymptotics of another class of maps in arbitrary surfaces. I'll focus on some more general questions.

One would expect almost all maps in almost any class to be asymmetric. If this were true, it would lead to a simple asymptotic relationship between maps and rooted maps since there are $4n$ ways to root an asymmetric map. Unfortunately, this "obvious" asymmetry has so far been difficult to establish for even the simplest classes of maps. Two results in this direction for the sphere are [9] and [17].

**Problem 7.** *Develop some general tools or results that can be used to conclude that almost all maps in various classes in an arbitrary surface are asymmetric.*

| |
|---|
| I misjudged this one. Richmond and Wormald [18] found an elegant result based on the fact that a typical map has many copies of any given submap. |

The previous problem is probably quite difficult, so perhaps one should (temporarily?) settle for a (still probably difficult) special case such as

**Problem 8.** *Prove that almost all maps in some particular class in an arbitrary surface are asymmetric.*

| |
|---|
| In addition to [18], see the results [6] for some enumeration by vertices and faces. |

It is not clear what the easiest class would be. Connectivity might help establish asymmetry, so something like 3-connected maps might be easier to deal with than all maps.

The asymptotic study of various classes of maps in arbitrary surfaces, beginning with [3], has led to the unfolding of a remarkable pattern in the asymptotics: let $M_S(n)$ be the number of $n$-edged rooted maps in some class $C$ in the surface $S$. Then, for those classes of maps which have so far been enumerated asymptotically on arbitrary surfaces,

$$M_S(n) \sim \alpha(S,C) n^{-5\chi(S)/4} \beta(C)^n, \tag{4.1}$$

where $\chi(S)$ is the Euler characteristic of $S$ and $\beta$ is independent of $S$. In addition, Gao has discovered that $\alpha$ has a remarkable structure. For a fuller discussion of the pattern in (4.1), see [14]. Of course, the possibility always exists that this pattern is an artifact: those maps which were amenable to study have a particular structure to their generating functions which makes them amenable. I believe that the pattern holds much more generally than has been established. One of the simplest problems suggested by (4.1) is

**Problem 9.** *Show that for any "reasonably defined" (whatever that means) class $C$ of maps, we have*

$$0 < \lim_{n\to\infty} (M_S(n))^{1/n} = \beta(C) < \infty;$$

*that is, the limit exists, is not trivial, and is independent of $S$.*
**Added***: There are classes that don't follow the pattern; e.g., maps with only one vertex or maps with a global symmetry, so "reasonably defined" should rule them out.* Of course, more difficult problems suggest themselves, such as

**Problem 10.** *"Explain" the presence of the Euler characteristic in (4.1).*

## References

1 D. Arquès and A. Giorgetti, Énumération des cartes pointées sur une surface orientable de genre quelconque en fonction des nombres de sommets et de faces, *J. Combin. Theory Ser. B* **77** (1999) 1–24.

2. E.A. Bender, The number of convex polyhedra, *Amer. Math. Monthly* **94** (1987) 7–21.

3. E.A. Bender and E.R. Canfield, The asymptotic number of rooted maps on a surface, *J. Combin. Theory Ser. A* **43** (1986) 244–257.

4. E.A. Bender and E.R. Canfield, The number of rooted maps on an orientable surface, *J. Combin. Theory Ser. B* **53** (1991) 293–299.

5. E.A. Bender and E.R. Canfield, The number of degree restricted rooted maps on the sphere, *SIAM J. Discrete Math.* **7** (1994) 9–15.

6. E.A. Bender, E.R. Canfield, Z.C. Gao, and L.B. Richmond, Submap density and asymmetry results for two parameter map families, *Combin. Probab. and Comput.* **6** (1997) 17–25.

7. E.A. Bender and L.B. Richmond, A survey of the asymptotic behaviour of maps, *J. Combin. Theory Ser. B* **40** (1986) 297–329.

8. E.A. Bender and N.C. Wormald, The number of rooted convex polyhedra, *Canad. Math. Bull.* **31** (1988) 99-102.

9. E.A. Bender and N.C. Wormald, Almost all convex polyhedra are asymmetric, *Canad. J. Math.* **27** (1985) 854–871.

10. D.R.L. Brown and D.M.Jackson, The quadrangulation conjecture for orientable surfaces, *J. Combin. Theory Ser. B* **86** (2002) 54–79.

11. W.G. Brown, On the existence of square roots in certain rings of power series, *Math. Annalen* **158** (1965) 82–89.

12. W.G. Brown, On the enumeration of non-planar maps, *Mem. Amer. Math. Soc.* **65** (1966).

13. E.R. Canfield, Calculating the number of rooted maps on surfaces, *Congressus Numerantium* **76** (1990) 21–34.

14. Z.C. Gao, A pattern for the asymptotic number of rooted maps on surfaces, *J. Combin. Theory Ser. A* **64** (1993) 246–264.

15. D.M. Jackson and T.I. Visentin, A character theoretic approach to embeddings of rooted maps in an orientable surface of given genus, *Trans. Amer. Math. Soc.* **322** (1990) 343–363.

16. R.C. Mullin and P.J. Schellenberg, The enumeration of $c$-nets via quadrangulations, *J. Combin. Theory* **4** (1968) 259–276.

17. L.B. Richmond and N.C. Wormald, Almost all quadrangular dissections of the disc are asymmetrical, *Math. Chronicle* **19** (1990) 63–71.

18. L.B. Richmond and N.C. Wormald, Almost all maps are asymmetric, *J. Combin. Theory Ser. B* **63** (1995) 1–7.

20. T.R.S. Walsh and A.B. Lehman, Counting rooted maps by genus. I–III, *J. Combin. Theory Ser. B* **13** (1972) 192-218, 122–141, **18** (1975) 222–259.