# The map asymptotics constant $t_g$

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**Abstract**

The constant $t_g$ appears in the asymptotic formulas for a variety of rooted maps on the orientable surface of genus $g$. Heretofore, studying this constant has been difficult. A new recursion derived by Goulden and Jackson for rooted cubic maps provides a much simpler recursion for $t_g$ that leads to estimates for its asymptotics.

# 1 Introduction

Let $\Sigma_g$ be the orientable surface of genus $g$. A *map* on $\Sigma_g$ is a graph $G$ embedded on $\Sigma_g$ such that all components of $\Sigma_g - G$ are simply connected regions. These components are called *faces* of the map. A map is rooted by distinguishing an edge, an end vertex of the edge and a side of the edge.

With $M_{n,g}$ the number of rooted maps on $\Sigma_g$ with $n$ edges, Bender and Canfield [1] showed that

$$M_{n,g} \sim t_g n^{5(g-1)/2} 12^n \quad \text{as} \quad n \to \infty, \tag{1}$$

where the $t_g$ are positive constants which can be calculated recursively using a complicated recursion involving, in addition to $g$, many other parameters. The first three values are

$$t_0 = \frac{2}{\sqrt{\pi}}, \quad t_1 = \frac{1}{24} \quad \text{and} \quad t_2 = \frac{7}{4320\sqrt{\pi}}.$$

Gao [3] showed that many other interesting families of maps also satisfy asymptotic formulas of the form

$$\alpha t_g (\beta n)^{5(g-1)/2} \gamma^n \tag{2}$$

and presented a table of $\alpha$, $\beta$ and $\gamma$ for eleven families. Richmond and Wormald [6] showed that many families of unrooted maps have asymptotics that differ from the rooted asymptotics by a factor of four times the number of edges. See Goulden and Jackson [4] for a discussion of connections with mathematical physics.

Although $\alpha$, $\beta$ and $\gamma$ in (2) seem relatively easy to compute, the common factor $t_g$ has been difficult to study. A recursion for rooted "cubic" maps derived by Goulden and Jackson [4] leads to a much simpler recursion for $t_g$ than that in [1]. We will use it to derive the following recursion and asymptotic estimate for $t_g$.

**Theorem 1** *Define $u_g$ by $u_1 = 1/10$ and*

$$u_g = u_{g-1} + \sum_{h=1}^{g-1} \frac{1}{R_1(g,h) R_2(g,h)} u_h u_{g-h} \quad \textit{for} \quad g \geq 2, \tag{3}$$

*where*

$$R_1(g,h) = \frac{[1/5]_g}{[1/5]_h [1/5]_{g-h}}, \quad R_2(g,h) = \frac{[4/5]_{g-1}}{[4/5]_{h-1}[4/5]_{g-h-1}}$$

*and* $[x]_k$ *is the rising factorial* $x(x+1)\cdots(x+k-1)$*. Then*

$$\begin{aligned} t_g &= 8\frac{[1/5]_g[4/5]_{g-1}}{\Gamma\left(\frac{5g-1}{2}\right)}\left(\frac{25}{96}\right)^g u_g \\ &\sim \frac{40\sin(\pi/5)K}{\sqrt{2\pi}}\left(\frac{1440g}{e}\right)^{-g/2} \quad as \quad g\to\infty, \end{aligned} \tag{4}$$

*where* $u_g \sim K \doteq 0.1049$ *is a constant.*
**Added after publication***: Marino [5] has pointed out that*

$$K = \frac{(3/5)^{1/2}\,\Gamma(1/5)\,\Gamma(4/5)}{4\pi^2}.$$

## 2 Cubic Maps

A map is called cubic if all its vertices have degree 3. The dual of cubic maps are called triangular maps whose faces all have degree 3. Let $T_{n,g}$ be the number of triangular maps on $\Sigma_g$ with $n$ vertices and let $C_{n,g}$ be the number of cubic maps on $\Sigma_g$ with $2n$ vertices. It was shown in [2] that

$$T_{n,g} \sim 3\left(3^7\times 2^9\right)^{(g-1)/2} t_g n^{5(g-1)/2}(12\sqrt{3})^n \quad \text{as} \quad n\to\infty. \tag{5}$$

Since a triangular map on $\Sigma_g$ with $v$ vertices has exactly $2(v+2g-2)$ faces,

$$C_{n,g} = T_{n-2g+2,g} \sim 3\times 6^{(g-1)/2} t_g n^{5(g-1)/2}(12\sqrt{3})^n \quad \text{as} \quad n\to\infty. \tag{6}$$

Define

$$H_{n,g} = (3n+2)C_{n,g} \quad \text{for} \quad n\geq 1, \tag{7}$$

$$H_{-1,0}=1/2, \quad H_{0,0}=2 \quad \text{and} \quad H_{-1,g}=H_{0,g}=0 \;\; \text{for} \;\; g\neq 0.$$

Goulden and Jackson [4] derived the following recursion for $(n,g)\neq(-1,0)$:

$$H_{n,g} = \frac{4(3n+2)}{n+1}\left(n(3n-2)H_{n-2,g-1} + \sum_{i=-1}^{n-1}\sum_{h=0}^{g} H_{i,h}H_{n-2-i,g-h}\right). \tag{8}$$

This is significantly simpler than the recursion derived in [2]. We will use it to derive information about $t_g$.

## 3 Generating Functions

Define the generating functions

$$T_g(x)=\sum_{n\geq 0}T_{n,g}x^n, \quad C_g(x)=\sum_{n\geq 0}C_{n,g}x^n, \quad H_g(x)=\sum_{n\geq 0}H_{n,g}x^n \quad \text{and} \quad F_g(x)=x^2H_g(x).$$

It was shown in [2] that $T_g(x)$ is algebraic for each $g \geq 0$, and

$$T_0(x) \;=\; \frac{1}{2}t^3(1-t)(1-4t+2t^2) \quad \text{with} \quad x \;=\; \frac{1}{2}t(1-t)(1-2t), \tag{9}$$

where $t = t(x)$ is a power series in $x$ with non-negative coefficients.

It follows from (6) and (7) that

$$\begin{aligned} C_g(x) &= x^{2g-2}T_g(x) \quad \text{for} \quad g \geq 0, &(10)\\ F_g(x) &= 3x^3C_g'(x) + 2x^2C_g(x) \quad \text{for} \quad g \geq 1. &(11)\end{aligned}$$

We also have

$$\begin{aligned} F_0(x) &= H_{0,0}x^2 + \sum_{n\geq 1}(3n+2)C_{n,0}x^{n+2} \\ &= 2x^2 + 3x^3C_0'(x) + 2x^2C_0(x) \\ &= 2x^2 + 3xT_0'(x) - 4T_0(x) \\ &= \frac{1}{2}t^2(1-t), \end{aligned} \tag{12}$$

where we have used (9). Hence $C_g(x)$ and $F_g(x)$ are both algebraic for all $g \geq 0$.

In the following we assume $g \geq 1$. From the recursion (8), we have

$$\begin{aligned} \frac{1}{4}\sum_{n\geq 0}\frac{n+1}{3n+2}H_{n,g}x^n &= \sum_{n\geq 1} n(3n-2)H_{n-2,g-1}x^n \\ &\quad + 2\sum_{n\geq 0}H_{-1,0}H_{n-1,g}x^n + x^2\sum_{h=0}^{g}H_h(x)H_{g-h}(x). \end{aligned}$$

Using (7) with a bit manipulation, we can rewrite the above equation as

$$\begin{aligned} \frac{1}{4}\sum_{n\geq 0}(n+1)C_{n,g}x^n &= 3x^2F_{g-1}''(x) + xF_{g-1}'(x) + xH_{-1,g-1} \\ &\quad + x^{-1}F_g(x) + x^{-2}\sum_{h=0}^{g}F_h(x)F_{g-h}(x). \end{aligned}$$

With $\delta_{i,j}$ the Kronecker delta, this becomes

$$\begin{aligned} x^3C_g'(x) + x^2C_g(x) &= 12x^4F_{g-1}''(x) + 4x^3F_{g-1}'(x) + 2x^3\delta_{g,1} \\ &\quad + 4xF_g(x) + 8F_0(x)F_g(x) + 4\sum_{h=1}^{g-1}F_h(x)F_{g-h}(x). \end{aligned}$$

It follows from (11) that

$$\begin{aligned} \left(1 - 12x - 24F_0(x)\right)F_g(x) &= 36x^4F_{g-1}''(x) + 12x^3F_{g-1}'(x) + 6x^3\delta_{g,1} \\ &\quad + 12\sum_{h=1}^{g-1}F_h(x)F_{g-h}(x) - x^2C_g(x). \end{aligned} \tag{13}$$

Substituting (12) and (9) into (13), we obtain

$$F_g(x) = \frac{1}{1-6t+6t^2}\Big(36x^4F''_{g-1}(x) + 12x^3F'_{g-1}(x) + 6x^3\delta_{g,1} + 12\sum_{h=1}^{g-1} F_h(x)F_{g-h}(x) - x^2C_g(x)\Big). \tag{14}$$

We now show that this equation can be used to calculate $C_g(x)$ more easily than the method in [2]. For this purpose we set $s = 1-6t+6t^2$ and show inductively that $C_g(x)$ is a polynomial in $s$ divided by $s^a$ for some integer $a = a(g) > 0$. (It can be shown that $a = 5g-3$ is the smallest such $a$, but we do not do so.) The method for calculating $C_g(x)$ follows from the proof. Then we have

$$x^2 = \frac{1}{432}(s-1)^2(2s+1) \qquad \text{and} \qquad \frac{ds}{dx} = \frac{144x}{s(s-1)}. \tag{15}$$

Thus

$$x\frac{d}{dx} = x\frac{ds}{dx}\frac{d}{ds} = \frac{(s-1)(2s+1)}{3s}\frac{d}{ds},$$

$$\frac{d^2}{dx^2} = \left(\frac{ds}{dx}\right)^2\frac{d^2}{ds^2} + \frac{d(ds/dx)}{dx}\frac{d}{ds} = \frac{48(2s+1)}{s^2}\frac{d^2}{ds^2} - \frac{48(s+1)}{s^3}\frac{d}{ds}.$$

From the above and (11)

$$F_g(x) + \frac{x^2C_g}{1-6t+6t^2} = x^2\left(3x\frac{dC_g}{dx} + \frac{(2s+1)C_g}{s}\right) = \frac{x^2(2s+1)}{s}\frac{d((s-1)C_g)}{ds}.$$

With some algebra, (14) can be rewritten as

$$\frac{d((s-1)C_g)}{ds} = \frac{4(s-1)^2(2s+1)}{s^2}\frac{d^2F_{g-1}}{ds^2} + \frac{4(s-1)}{s^3}\frac{dF_{g-1}}{ds} + \frac{5184}{(s-1)^2(2s+1)^2}\sum_{h=1}^{g-1}F_hF_{g-h} \qquad \text{for } g \ge 2. \tag{16}$$

In what follows $P(s)$ stands for a polynomial in $s$ and $a$ a positive integer, both different at each occurrence. It was shown in [2] that

$$C_1(x) = T_1(x) = \frac{1-s}{12s^2}.$$

By (11), (15) and the induction hypothesis, the right hand side of (16) has the form $P(s)/s^a$. Integrating, $(s-1)C_g = P(s)/s^a + K\log s$. Since we know $C_g(x)$ is algebraic, so is $(s-1)C_g$ and hence $K=0$. Since $s=1$ corresponds to $x=0$, $C_g$ is defined there. It follows that $P(s)$ in $(s-1)C_g = P(s)/s^a$ is divisible by $s-1$, completing the proof.

Using Maple, we obtained

$$
\begin{aligned}
C_2 &= \frac{1}{2^6\,3^4}\frac{(2s+1)(17s^2+60s+28)(1-s)^3}{s^7},\\
C_3 &= \frac{1}{2^9\,3^8}\frac{(5052s^4-747s^3-33960s^2-35620s-9800)(2s+1)^2(s-1)^5}{s^{12}},\\
C_4 &= \frac{1}{2^{14}\,3^{11}}\frac{P_4(s)(2s+1)^3(s-1)^7}{s^{17}},\\
C_5 &= \frac{1}{2^{17}\,3^{14}}\frac{P_5(s)(2s+1)^4(1-s)^9}{s^{22}},
\end{aligned}
$$

where

$$
\begin{aligned}
P_4(s) &= -12458544-63378560s-103689240s^2-42864016s^3\\
&\quad +31477893s^4+20750256s^5+417636s^6,\\
P_5(s) &= 7703740800+50294009360s+117178660480s^2\\
&\quad +100386081272s^3-16827627792s^4-67700509763s^5\\
&\quad -21455389524s^6+4711813020s^7+1394857272s^8.
\end{aligned}
$$

# 4 Generating Function Asymptotics

Suppose $A(x)$ is an algebraic function and has the following asymptotic expansion around its dominant singularity $1/r$:

$$
A(x) \;=\; \sum_{j=l}^{k} a_j(1-rx)^{j/2} + O\left((1-rx)^{(k+1)/2}\right),
$$

where $a_j$ are not all zero. Then we write

$$
A(x) \;\approx\; \sum_{j=l}^{k} a_j(1-rx)^{j/2}.
$$

The following lemma is proved in [2].

**Lemma 1** *For $g \geq 0$, $T_g(x)$ is algebraic,*

$$
\begin{aligned}
T_0(x) &\approx \frac{\sqrt{3}}{72}-\frac{5}{216}+\frac{1}{54\sqrt{6}}(1-12\sqrt{3}x)^{3/2},\\
T_g(x) &\approx 3\left(3^7\times 2^9\right)^{(g-1)/2} t_g\,\Gamma\!\left(\frac{5g-3}{2}\right)(1-12\sqrt{3}x)^{-(5g-3)/2} \quad \textit{for} \quad g\geq 1.
\end{aligned}
$$

Let

$$f_g \;=\; 24^{-3/2}6^{g/2}\,\Gamma\!\left(\frac{5g-1}{2}\right)t_g. \tag{17}$$

Using Lemma 1, (10) and (11), we obtain

$$\begin{aligned}C_g(x) &\approx \frac{288}{(5g-3)}f_g(1-12\sqrt{3}x)^{-(5g-3)/2} \quad \text{for} \quad g\ge 1,\\ F_g(x) &\approx f_g(1-12\sqrt{3}x)^{-(5g-1)/2} \quad \text{for} \quad g\ge 1.\end{aligned}$$

As noted in [2], the function $t(x)$ of (9) has the following asymptotic expansion around its dominant singularity $x=\frac{1}{12\sqrt{3}}$:

$$t \;\approx\; \frac{3-\sqrt{3}}{6}-\frac{\sqrt{2}}{6}(1-12\sqrt{3}x)^{1/2}.$$

Using this and (12), we obtain

$$\begin{aligned}F_0(x) &\approx \frac{3-\sqrt{3}}{72}+f_0(1-12\sqrt{3}x)^{1/2},\\ \frac{1}{1-6t+6t^2} &\approx \frac{\sqrt{6}}{2}(1-12\sqrt{3}x)^{-1/2}.\end{aligned}$$

Comparing the coefficients of $(1-12\sqrt{3}x)^{(5g-1)/2}$ on both sides of (14), we obtain

$$f_g \;=\; \frac{\sqrt{6}}{96}(5g-4)(5g-6)f_{g-1}+6\sqrt{6}\sum_{h=1}^{g-1}f_hf_{g-h}. \tag{18}$$

Letting

$$u_g \;=\; f_g\left(\frac{25\sqrt{6}}{96}\right)^{-g}\frac{6\sqrt{6}}{[1/5]_g[4/5]_{g-1}}.$$

and using (17), the recursion (18) becomes (3).

# 5 Asymptotics of $t_g$

It follows immediately from (3) that $u_g\ge u_{g-1}$ for all $g\ge 2$. To show that $u_g$ approaches a limit $K$ as $g\to\infty$, it suffices to show that $u_g$ is bounded above. The value of $K$ is then calculated using (3).

We use induction to prove $u_g\le 1$ for all $g\ge 1$. Since $u_1=\frac{1}{10}$ and $u_2=u_1+\frac{1}{480}$, we can assume $g\ge 3$ for the induction step. From now on $g\ge 3$.

Note that

$$\begin{aligned}R_1(g,1)R_2(g,1) &= 5(g-\tfrac{4}{5})(g-\tfrac{6}{5}) \;>\; 5(g-\tfrac{4}{5})(g-\tfrac{9}{5})\\ R_1(g,2)R_2(g,2) &= \tfrac{25}{24}(g-\tfrac{6}{5})(g-\tfrac{11}{5})\Big(5(g-\tfrac{4}{5})(g-\tfrac{9}{5})\Big)\\ &> \tfrac{25}{24}(g-\tfrac{6}{5}+\tfrac{4}{5})(g-\tfrac{11}{5}-\tfrac{4}{5})\Big(5(g-\tfrac{4}{5})(g-\tfrac{9}{5})\Big)\\ &\ge 2(g-3)\Big(5(g-\tfrac{4}{5})(g-\tfrac{9}{5})\Big).\end{aligned}$$

Note that $R_i(g,h) = R_i(g,g-h)$ and, for $h < g/2$, $\frac{R_i(g,h+1)}{R_i(g,h)} \geq 1$. Combining all these observations and the induction hypothesis with (3) we have

$$\begin{aligned}
u_g &= u_{g-1} + \sum_{h=1}^{g-1} \frac{u_h u_{g-h}}{R_1(g,h)R_2(g,h)} \\
&< u_{g-1} + \frac{2u_1 u_{g-1}}{5(g-\frac{4}{5})(g-\frac{9}{5})} + \sum_{h=2}^{g-2} \frac{1}{R_1(g,2)R_2(g,2)} \\
&< u_{g-1} + \frac{1/5}{5(g-\frac{4}{5})(g-\frac{9}{5})} + \frac{1/2}{5(g-\frac{4}{5})(g-\frac{9}{5})} \\
&< u_{g-1} + \frac{1}{5g-9} - \frac{1}{5g-4}.
\end{aligned}$$

Hence

$$u_g \;<\; u_2 + \sum_{k=3}^{g} \left( \frac{1}{5k-9} - \frac{1}{5k-4} \right) \;<\; u_2 + \frac{1}{5 \times 3 - 9} \;<\; 1.$$

The asymptotic expression for $t_g$ in (4) is obtained by using

$$[x]_k = \frac{\Gamma(x+k)}{\Gamma(x)}, \qquad \Gamma(1/5)\Gamma(4/5) = \frac{\pi}{\sin(\pi/5)},$$

and Stirling's formula

$$\Gamma(ag+b) \sim \sqrt{2\pi}(ag)^{b-1/2} \left(\frac{ag}{e}\right)^{ag} \quad \text{as} \quad g \to \infty,$$

for constants $a > 0$ and $b$.

# 6 Open Questions

We list some open questions.

- From (18), we can show that $f(z) = \sum_{g \geq 1} f_g z^g$ satisfies the following differential equation

  $$f(z) = 6\sqrt{6}(f(z))^2 + \frac{\sqrt{6}}{96} z \left( 25z^2 f''(z) + 25zf'(z) - f(z) + \frac{\sqrt{6}}{72} \right).$$

  The asymptotic expression of $f_g$ implies that $f(z)$ cannot be algebraic. Can one show that $f(z)$ is not D-finite, that is, $f(z)$ does not satisfy a linear differential equation?

- There is a constant $p_g$ that plays a role for maps on non-orientable like $t_g$ plays for maps on orientable surfaces [3]. Is there a recursion for maps on non-orientable surfaces that can be used to derive a theorem akin to Theorem 1 for $p_g$?

- Find simple recursions akin to (8) for other classes of rooted maps that lead to simple recursive calculations of their generating functions as in (16).

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