# A Discontinuity in the Distribution of Fixed Point Sums

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AMS Subject Classification: 05A17, 05A20, 05A16, 11P81

Submitted: October 19, 2002; Accepted: March 29, 2003.

**Abstract**

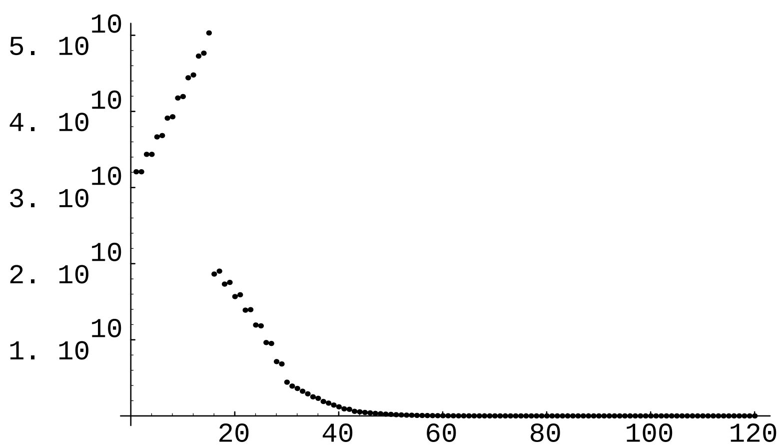
The quantity $f(n,r)$, defined as the number of permutations of the set $[n] = \{1,2,\ldots n\}$ whose fixed points sum to $r$, shows a sharp discontinuity in the neighborhood of $r = n$. We explain this discontinuity and study the possible existence of other discontinuities in $f(n,r)$ for permutations. We generalize our results to other families of structures that exhibit the same kind of discontinuities, by studying $f(n,r)$ when "fixed points" is replaced by "components of size 1" in a suitable graph of the structure. Among the objects considered are permutations, all functions and set partitions.

# 1 Introduction

Let $f(n,r)$ denote the number of permutations of $[n] = \{1,2,\ldots,n\}$, the sum of whose fixed points is $r$. For example, when $n = 3$ we find the values

| $r$ | 0 | 1 | 2 | 3 | 6 |
|---|---|---|---|---|---|
| $f(3,r)$ | 2 | 1 | 1 | 1 | 1 |

Here is the graph of $f(15,r)$:




The plot shows an interesting steep drop from $r = n$ to $r = n+1$, and this paper arose in providing a quantification of the observed plunge. We think that this problem is a nice example of an innocent-looking asymptotic enumerative situation in which thoughts of discontinuities might be far from the mind of an investigator, yet they materialize in an interesting and important way. A quick explanation for the discontinuity is as follows: about 74% of permutations have fewer than two fixed points, and of course with only one or no fixed points the sum cannot exceed $n$.

Given this explanation for the discontinuity at $r = n$, it seems reasonable to expect further discontinuities. For example, when $r = n + (n-1)$ two fixed points suffice, but $r = 2n$ requires at least three fixed points. Thus, the lack of further discontinuities in the graph of $f(15, r)$ may, at first, seem counterintuitive. We discuss it in the next section.

To explore the presence of this gap behavior in other situations, we require some terminology.

**Definition 1** *For each $n > 0$, let $\mathcal{G}_n$ be a set of structures of some sort containing $n$ points whose labels are the set $[n] = \{1, 2, \ldots, n\}$. We call them* labeled structures. *Let $G_n = |\mathcal{G}_n|$ and, for convenience, $G_0 = 1$.*

*Suppose the notion of* fixed *is defined for points in these structures.*

- *Let $\mathcal{D}_n$ be the elements of $\mathcal{G}_n$ without fixed points and let $D_n = |\mathcal{D}_n|$.*
- *If $\mathcal{K} \subseteq [n]$, let $G_{n,\mathcal{K}}$ be the number of structures in $\mathcal{G}_n$ whose fixed points have exactly the labels $\mathcal{K}$ and let $G_{n,k}$ be the number of structures having exactly $k$ fixed points. Thus $D_n = G_{n,0}$.*
- *For each integer $r$, let $f(n, r)$ be the number of structures in $\mathcal{G}_n$ such that the sum of the labels of its fixed points equals $r$.*

We can roughly describe when the gap at $r = n$ will occur. Suppose $G_n$, $D_n$ and the way labels can be used are reasonably well behaved. Here are two descriptions of when we can expect the gap to occur.

(i) There is a gap if and only if the exponential generating function $G(x) = \sum G_n x^n/n!$ has radius of convergence $\rho < \infty$. In this case $f(n, r) \sim \phi(r/n)(G_n - D_n)/n$, where $\phi$ is a function that has discontinuities only at 0 and 1. Furthermore, if $\rho = 0$, then $\phi$ is the characteristic function of the interval $(0, 1]$.

(ii) Let $X_n$ be a random variable equal to the sum of the fixed points in an element of $\mathcal{G}_n \setminus \mathcal{D}_n$ chosen uniformly at random. There is a gap if and only if the expected value of $X_n$ grows linearly with $n$. In this case, $n\,\mathrm{Prob}(X_n = r) \sim \phi(r/n)$, where $\phi$ is the function in (i).

Our focus will be on the existence of a nontrivial gap; i.e., $0 < \rho < \infty$. We phrase our results in terms of counts rather than probabilities.

**Definition 2** *Let $\mathcal{G}_n$ be a set of structures as in Definition 1. We call the $\mathcal{G}_n$ a* Poisson family *with parameters $0 < C < \infty$ and $0 < \lambda < \infty$ if the following three conditions hold.*

*(a) For each $k$ we have $G_{n,\mathcal{K}} \sim C^k D_{n-k}$ uniformly as $n \to \infty$ with $\mathcal{K} \subseteq [n]$ and $|\mathcal{K}| = k$.*

*(b)* $\limsup_{n\to\infty} \max_{0\le m\le n} \left( \left( \frac{G_m/m!}{G_n/n!} \right)^{n-m} \right) < \infty$

*(c) For each fixed $k \ge 0$,* $G_{n,k} = G_n\Big(e^{-\lambda}\lambda^k/k! + o(1)\Big)$ *as $n \to \infty$.*

**Remark 1** The meaning of the statement that $A(n,x) \to 0$ uniformly as $n \to \infty$ with $x \in \mathcal{R}_n$ is that

$$\lim_{n\to\infty} \sup_{x\in\mathcal{R}_n} |A(n,x)| = 0.$$

Thus in order to bring our objects of study (Poisson families) into the scope of Definition 2(a), we need only to take $\mathcal{R}_n$ be the set of $k$-subsets of $[n]$. □

**Remark 2** To understand the definition better, we look at how it applies when $\mathcal{G}_n$ is the set of permutations of $[n]$ and fixed points have their usual meaning.

- $D_n$ is the number of derangements of $[n]$.
- With $C = 1$, (a) is in fact an equality for all $\mathcal{K} \subseteq [n]$. It follows from the fact that the permutations of $[n]$ with fixed points $\mathcal{K}$ are the derangements of $[n] \setminus \mathcal{K}$.
- Since $G_n = n!$, (b) holds.
- For (c), first recall $D_n \sim n!/e$, as proven, for example, in [2] and on page 144 of [10]. Then note that $G_{n,k} = \binom{n}{k} D_{n-k}$ since we choose $k$ fixed points and derange the rest. Hence
$$G_{n,k} \sim \frac{n!}{k!\,(n-k)!}\,\frac{(n-k)!}{e} = \frac{G_n}{e\,k!}$$
and so $\lambda = 1$.

The fact that permutations are a Poisson family also follows easily from the next lemma with $C = \rho = 1$. □

**Lemma 1** *Given a family of labeled structures, suppose that for some $0 < \rho, C < \infty$ the following hold.*

*(i) For all $\mathcal{K} \subset [n]$ we have $G_{n,\mathcal{K}} = C^{|\mathcal{K}|} D_{n-|\mathcal{K}|}$.*

*(ii) $nG_{n-1}/G_n \sim \rho$.*

*Then the family of structures is Poisson with parameters $C$ and $\lambda = C\rho$.*

**Remark 3** The conditions deserve some comment. Suppose there are $C$ types of fixed points. Condition (i) follows if there is no constraint on structures based on labels and one can build structures with fixed points $S$ by

(a) choosing independently a type for each fixed point,

(b) choosing any fixed-point free structure $D$ on $n-|S|$ labels for the rest of the structure and

(c) replacing $i$ by $a_i$ in $D$ where $[n] \setminus S \ = \ \{a_1 < a_2 < \ldots\}$.

At first, deciding the truth or falsity of (i) in a particular instance may seem trivial. Not so, however.

- Permutations in which elements in the same cycle must have the same parity have a label-based constraint on structures. Hence the replacement in (c) may not give a valid structure.
- Permutations with an *odd* number of cycles violate (b) because the parity of the number of cycles in the structure $D$ chosen there must be opposite the parity of $|S|$ so that the final structure will have an odd number of cycles. See Example 4 below for further discussion.

Condition (ii) merely asserts that the exponential generating function for $G_n$ has radius of convergence $\rho$ and that the $G_n$ grow smoothly. Since $\rho$ is the radius of convergence and we assumed $0 < \rho < \infty$, the lemma does not apply to entire functions or to purely formal power series. In those cases, if the $G_n$ are well behaved the situation is, in a sense, like having $\lambda = \infty$ and $\lambda = 0$, respectively. We will discuss this further in the examples. □

Let

$$\chi(\text{statement}) \ = \ \begin{cases} 1, & \text{if statement is true,} \\ 0, & \text{if statement is false.} \end{cases}$$

Recall that $f(n,r)$ is the number of labeled structures in $\mathcal{G}_n$ for which the labels of the fixed points sum to $r$.

**Theorem 1** *If $\mathcal{G}_n$ is a Poisson family of structures with parameters $C$ and $\lambda$, then the graph of $f(n,r)$, appropriately scaled, exhibits one and only one gap as $n \to \infty$ and that at $r = n$. More precisely, there is a continuous strictly decreasing function $K_\mu$ with domain $(0, \infty)$ such that, for any constants $0 < a < b < \infty$,*

$$f(n,r) \ = \ D_{n-1}\Big(K_{\lambda/C}(r/n) + \chi(r \le n) + o(1)\Big) \tag{1.1}$$

*uniformly as $n \to \infty$ is such a way that $r = r(n)$ satisfies $a \le r/n \le b$. In fact,*

$$K_\mu(\alpha) \ = \ \sum_{k=2}^{\infty} \frac{c_k(\alpha)(\alpha\mu)^{k-1}}{k!\,(k-1)!}$$

*where, for $k \geq 2$, $c_k$ is the decreasing continuous function*

$$c_k(\alpha) \;=\; \sum_{0\leq j<\alpha} \binom{k}{j}(-1)^j\left(1-\frac{j}{\alpha}\right)^{k-1}$$

*having domain $(0,\infty)$, codomain $[0,1]$ and support $(0,k)$.*

Here is a small table of $K_\mu(\alpha)$, rounded in the fifth decimal place.

| $\alpha$ | $K_1(\alpha)$ | $K_{1/e}(\alpha)$ | $K_{1/2e}(\alpha)$ |
|---|---|---|---|
| 0.5 | 0.27172 | 0.09483 | 0.04670 |
| 1.0 | 0.59064 | 0.19557 | 0.09483 |
| 1.5 | 0.39670 | 0.10991 | 0.05034 |
| 2.0 | 0.11525 | 0.01273 | 0.00300 |
| 2.5 | 0.04645 | 0.00391 | 0.00083 |
| 3.0 | 0.01162 | 0.00042 | 0.00005 |
| 3.5 | 0.00324 | 0.00008 | 0.00001 |
| 4.0 | 0.00074 | 0.00001 | 0.00000 |

# 2 Some Examples

In this section, we look at some examples and at the question of why there is only one gap.

**Example 1** *All Permutations.* For permutations of $[n]$, $G_n = n!$. Apply Lemma 1 with $C = 1$ and $\rho = 1$ to see that Theorem 1 applies with $C = \lambda = 1$.

**Example 2** *All Functions.* Consider the set of all functions from $[n]$ to $[n]$ and call $x$ a fixed point when $f(x) = x$. Then $G_n = n^n$. We can apply Lemma 1 with $C = 1$ and $\rho = 1/e$ since

$$\lim_{n\to\infty}\frac{nG_{n-1}}{G_n} \;=\; \lim_{n\to\infty}\frac{n(n-1)^{n-1}}{n^n} \;=\; \lim_{n\to\infty}\frac{n}{n-1}\left(1-\frac{1}{n}\right)^n \;=\; 1/e.$$

Thus, Theorem 1 applies with $C = 1$ and $\lambda = 1/e$.

**Example 3** *Partial Functions.* A partial function $f$ from $[n]$ to $[n]$ is a function from a subset $\mathcal{D}$ of $[n]$ to $[n]$ and is undefined on $[n] \setminus \mathcal{D}$. The number of partial functions is $(n+1)^n$. Call $x$ a fixed point if either $f(x) = x$ or $f(x)$ is undefined. We can apply Lemma 1 with $C = 2$ and $\rho = 1/e$. The value $C = 2$ arises because there are two ways to make $x$ into a fixed point. The value of $\rho$ is found as was done for all functions. Hence $\lambda = 2/e$.

**Example 4** *Permutations with Restricted Cycle Lengths.* Consider permutations of $[n]$ with all cycle lengths odd. The exponential generating function for $G_n$ is $\sqrt{(1+x)/(1-x)}$. By Darboux's Theorem, $G_n \sim n!/\sqrt{\pi n/2}$. (For Darboux's Theorem see, for example, [2] or [10].) Thus Lemma 1 applies with $C = 1$ and $\rho = 1$.

**Example 5** *Labeled Forests of Rooted Trees.* Let $\mathcal{G}_n$ be the labeled forests on $[n]$ where each tree is rooted. Call a 1-vertex tree a fixed point. The number of labeled rooted trees is well known to be $n^{n-1}$. (See, for example, [2].) Since there are $n$ ways to root a labeled tree and since removing vertex $n$ from unrooted $n$-vertex trees gives a bijection with rooted $(n-1)$-vertex forests of rooted trees, there are $(n+1)^{n-1}$ $n$-vertex forests of rooted trees. We can apply Lemma 1 with $C = 1$ and $\rho = 1/e$.

**Example 6** *Labeled Forests of Unrooted Trees.* This is similar to the preceding example. The exact formula for the number of forests involves Hermite polynomials [9]; however, the asymptotics is simple [6]: $G_n \sim e^{1/2}n^{n-2}$. Thus we can apply Lemma 1 with $C = 1$ and $\rho = 1/e$.

**Example 7** *Permutations with a Restricted Number of Cycles.* Consider permutations of $[n]$ with an odd number of cycles. Although Lemma 1(i) fails, we claim this is a Poisson family and we may take $\lambda = 1$. It suffices to show that $G_n$, $G_{n,k}$, and $D_n$ are asymptotically half their values for all permutations. The generating function for all permutations, with $x$ keeping track of size, $y$ of fixed points and $z$ of number of cycles, is

$$P(x,y,z) \;=\; \exp\Bigl(xyz + \sum_{k=2}^{\infty} \frac{x^k z}{k!}\Bigr) \;=\; e^{x(y-1)z}(1-x)^{-z}.$$

By multisection of series, the generating function $G(x,y)$ for our present problem is

$$G(x,y) \;=\; \frac{P(x,y,1)}{2} \;-\; \frac{P(x,y,-1)}{2}.$$

The first term on the right is half the generating function for permutations and the last term is entire. Thus only the first term matters asymptotically. Hence we obtain asymptotic estimates for $G_n$ and $G_{n,k}$ that differ from those for all permutations by a factor of 2. Thus $C = 1$ and $\lambda = 1$. Other restrictions on number of cycles can often be handled in a similar manner. Similar results hold for functions with restrictions on the number of components in the associated functional digraphs.

What happens when Definition 2 fails because we would need $\lambda = 0$ or $\lambda = \infty$? Let $f(n)$ be the maximum of $f(n,r)$.

- When $\lambda = 0$, fixed points are rare. We can expect $f(n) = f(n,0)$ and, for each $r > 0$, $f(n,r) = o(f(n,0))$.

- When $\lambda = \infty$, fixed points are common. We can expect the $r$ for which $f(n) = f(n,r)$ to grow faster than $n$ and $f(n,r) = o(f(n))$ for $r = O(n)$.

Here are some examples.

**Example 8** *Involutions.* A permutation whose only cycle lengths are 1 and 2 is an *involution*. Let $\mathcal{G}_n$ be the involutions on $[n]$. Since $G_{n,\mathcal{K}} = D_{n-|\mathcal{K}|}$, condition (a) of Definition 2 trivially holds. The number of fixed-point free involutions is easily seen to be

$$D_n = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \dfrac{n!}{2^{n/2}(n/2)!}, & \text{if } n \text{ is even.} \end{cases}$$

Hence we have

$$G_{n,k} = \binom{n}{k} D_{n-k} = \begin{cases} 0, & \text{if } n-k \text{ is odd,} \\ \dbinom{n}{k} \dfrac{(n-k)!}{2^{(n-k)/2}\,((n-k)/2)!}, & \text{if } n-k \text{ is even.} \end{cases}$$

Using standard techniques for estimating sums, one obtains $\lim_{n\to\infty} G_{n,k}/G_n = 0$ for all fixed $k$ and so $\lambda = \infty$.

**Example 9** *Partitions of Sets.* Let $\mathcal{G}_n$ be the partitions of $[n]$ and let the fixed points be the blocks of size 1. Then $G_{n,\mathcal{K}} = D_{n-|\mathcal{K}|}$. Let $[x^n y^k]\, F(x,y)$ denote the coefficient of $x^n y^k$ in the generating function $F(x,y)$. It turns out that

$$G_{n,k} = n!\, [x^n y^k] \,\exp(e^x + xy - x - 1) = (n!/k!)\, [x^n]\, x^k \exp(e^x - x - 1).$$

Using methods as in Section 6.2 of [3], it can be shown that

$$[x^n]\, x^k \exp(e^x - x - 1) \sim u_n^k e^{-u_n}\, [x^n]\, \exp(e^x - 1),$$

where $u_n \sim \ln n$. Since $n!\, [x^n] \exp(e^x - 1) = G_n$, we again have $\lambda = \infty$.

**Example 10** *All Graphs.* Let $\mathcal{G}_n$ be all $n$-vertex labeled graphs. Then $G_n = 2^N$, where $N = \binom{n}{2}$ and $D_n \sim G_n$ because almost all graphs are connected. This is a $\lambda = 0$ situation. It turns out that $f(n,r) = o(f(n))$ for all $r$. The situation can be made more interesting by limiting our attention to graphs with $q(n)$ edges where $q(n)$ grows appropriately. If $n\, e^{-2q(n)/n} \to \lambda$ where $0 < \lambda < \infty$, then Definition 2(c) follows from [4]. Since $q(n) \sim (n \log n)/2$,

$$G_n = \binom{N}{q(n)} \approx \left(\frac{en}{\log n}\right)^{q(n)}.$$

Thus, as for all graphs, $\sum G_n x^n/n!$ has radius of convergence $\rho = 0$. One can show that the limsup in Definition 2(b) is zero. Definition 2(a) fails: one would need $C$ to be an unbounded function of $n$. The fact that, in a sense, $C \to \infty$ makes it possible to still prove (1.1); however, since $\lambda/C \to 0$, $K_{\lambda/C}$ becomes $K_0 \equiv 0$. All of this is typical of the situation where the structures are well behaved but $G_n$ grows too rapidly, except that one often has $\lambda = 0$.

Why is there only one gap? This is closely related to the fact that a fixed point has exactly one label.

A single label chosen at random is uniformly distributed on $[n]$, the set of possible labels. This leads to a discontinuity in the "sum" of a single label at $n$. The distribution of a sum of $k > 1$ labels chosen at random has a maximum near $kn/2$ and is much smaller near the extreme values the sum can achieve. Consider how the various $k$ contribute to $f(n, r)$. When the radius of convergence $\rho$ of $G(x)$ is between 0 and $\infty$, the contributions of the various $k$ scale in such a way that all contribute but the contribution falls off rapidly with increasing $k$, thus leading to a convergent series in which the $k = 1$ term is significant. When $\rho = 0$, the contribution of the various $k$ falls off more and more as $n \to \infty$ so that only $k = 1$ contributes in the limit. When $\rho = \infty$, the series is no longer convergent and so the discontinuity of $k = 1$, being of a lower order than the entire sum, vanishes in the limit.

What would happen if fixed points had more than one label? For example, suppose we perversely said that the fixed points of a permutation were the 2-cycles. Thus, a set of fixed points must have at least 2 labels and so we cannot have $k = 1$ in the preceding paragraph. Consequently, the discontinuity of $f(n, r)$ vanishes. On the other hand, if we had defined fixed points to be 1-cycles *and* 2-cycles, then $k = 1$ would be possible and so $f(n, r)$ would again have a gap at $r = n$.

# 3 Proof of Lemma 1

Clearly Lemma 1(i) is stronger than Definition 2(a).

From Lemma 1(ii), there is an $N$ such that

$$nG_{n-1}/G_n < 2\rho \quad \text{whenever} \quad n \geq N. \tag{3.1}$$

Note that $G_N \neq 0$. Let $A \geq \max(2\rho, 1)$ be such that $\frac{G_m/m!}{G_N/N!} < A^{N-m}$ whenever $m < N$. From (3.1), $\frac{G_m/m!}{G_n/n!} \leq (2\rho)^{n-m}$ whenever $n \geq N$ and $n > m \geq N$. Combining these two results gives $\frac{G_m/m!}{G_n/n!} \leq A^{n-m}$ whenever $n \geq N$ and $m \leq n$. This proves Definition 2(b).

Let $D(x) = \sum D_n x^n/n!$, $G(x) = \sum G_n x^n/n!$ and $G(x, y) = \sum G_{n,k} x^n y^k/n!$. From (i) we have $G_{n,k} = \binom{n}{k} C^k D_{n-k}$ since there are $\binom{n}{k}$ choices for $S$ with $|S| = k$. Thus

$$G(x, y) = \sum \frac{D_{n-k} x^{n-k} (Cxy)^k}{(n-k)!\, k!} = D(x) e^{Cxy}. \tag{3.2}$$

With $y = 1$, $D(x) = G(x)e^{-Cx}$.

From (3.2),

$$G_{n,k} \;=\; n!\,[x^n y^k]\,G(x,y) \;=\; n!\,\Big([x^{n-k}]\,D(x)\Big)\frac{C^k}{k!}. \tag{3.3}$$

From $D(x) = G(x)e^{-Cx}$, $[x^{n-k}]\,D(x) = [x^n]\,(G(x)\,x^k e^{-Cx})$. From Lemma 1(ii), $G(x)$ has radius of convergence $\rho$. Since $x^k e^{-Cx}$ is entire, it follows from Lemma 1(ii) and Schur's Lemma ([5], problem I.178) that $[x^{n-k}]\,D(x) \;\sim\; \rho^k e^{-C\rho}(G_n/n!)$. Substituting into (3.3), we have

$$G_{n,k} \;\sim\; \frac{(C\rho)^k e^{-C\rho} G_n}{k!}.$$

This completes the proof of Lemma 1. □

# 4 The General Plan

For simplicity in this overview we ignore questions of uniformity.

Given a set $\mathcal{K}$, let $\|\mathcal{K}\|$ denote the sum of the elements in $\mathcal{K}$. Let $E(r,k,n)$ be the number of $k$-subsets $\mathcal{K}$ of $[n]$ with $\|\mathcal{K}\| = r$. By definition,

$$f(n,r) \;=\; \sum_{\substack{\mathcal{K}\subseteq[n]\\ \|\mathcal{K}\|=r}} G_{n,\mathcal{K}} \;=\; \sum_{k\geq 1}\sum_{\substack{|\mathcal{K}|=k\\ \|\mathcal{K}\|=r}} G_{n,\mathcal{K}}.$$

By Definition 2(a), this becomes

$$f(n,r) \;\sim\; \sum_{k\geq 1} C^k D_{n-k} \sum_{\substack{|\mathcal{K}|=k\\ \|\mathcal{K}\|=r}} 1 \;=\; \sum_{k\geq 1} C^k D_{n-k} E(r,k,n).$$

A little thought shows that

$$E(r,1,n) \;=\; \begin{cases} 1, & \text{if } 0<r\leq n,\\ 0, & \text{otherwise.}\end{cases} \tag{4.1}$$

Thus

$$f(n,r) \;=\; D_{n-1}\left(\chi(0<r\leq n) + \sum_{k>1} E(r,k,n)\frac{D_{n-k}}{D_{n-1}}\right). \tag{4.2}$$

To use the sum (4.2) for asymptotics, we need estimates for $D_{n-k}/D_{n-1}$ and $E(r,k,n)$.

We begin with $D_{n-k}/D_{n-1}$. From Definition 2(a),

$$G_{n,t} \;\sim\; \binom{n}{t} C^t D_{n-t} \;\sim\; n^t C^t D_{n-t}/t!$$

and, from Definition 2(c),

$$G_{n,t} \;\sim\; G_n e^{-\lambda}\lambda^t/t!.$$

Combining these two, we have

$$D_{n-t} \ \sim \ e^{-\lambda}(\lambda/Cn)^t G_n. \tag{4.3}$$

With $t = k$ and $t = 1$ we obtain

$$\frac{D_{n-k}}{D_{n-1}} \ \sim \ (\lambda/Cn)^{k-1}. \tag{4.4}$$

Estimates for $E(r,k,n)$ are not so easy to come by for general values of the three parameters $(r,k,n)$. Szekeres [8] has obtained an asymptotic formula valid for $r \to \infty$ with $k, n$ in neighborhoods of their expected values $k_0(r), n_0(r)$. However, the range of use to us in this investigation is $n \to \infty$, $r/n$ bounded, and $k$ relatively small, say up to $n^\epsilon$. This is more easily handled than, but completely outside the range covered by, Szekeres' formula.

In the next section we study $c_k(\alpha)$, which plays a role in estimating $E(r,k,n)$. Asymptotics for $E(r,k,n)$ are established in Section 6. With this groundwork, it is a fairly simple matter to prove Theorem 1 in Section 7.

# 5 The Sum $c_k(\alpha)$

We recall the formula for $c_k(\alpha)$ from Theorem 1:

$$c_k(\alpha) \ = \ \sum_{0 \le j < \alpha} \binom{k}{j} (-1)^j \left(1 - \frac{j}{\alpha}\right)^{k-1}, \tag{5.1}$$

which we now take as a definition of $c_k(\alpha)$ for $k \ge 1$ and all $\alpha$.

**Lemma 2** *Let $c_k(\alpha)$ be given by (5.1) for $k \ge 1$. Then*

*(a) If $\alpha \le 0$ or $\alpha > k$, then $c_k(\alpha) = 0$.*

*(b) If $0 < \alpha \le 1$, then $c_k(\alpha) = 1$.*

*(c) $c_1(\alpha) = \chi(0 < \alpha \le 1)$.*

*(d) If $k \ge 2$, the function $c_k(\alpha)$ is continuous for $\alpha > 0$.*

*(e) If $k > \alpha \ge 1$, then $c_k(\alpha)$ is strictly decreasing and so $1 \ge c_k(\alpha) > 0$ for $0 < \alpha < k$.*

*Proof* When $\alpha \le 0$, the sum (5.1) is empty and so $c_k(\alpha) = 0$. Suppose $\alpha > k$. We have

$$c_k(\alpha) \ = \ \sum_{j=0}^{k} \binom{k}{j} (-1)^j \left(1 - \frac{j}{\alpha}\right)^{k-1} \ = \ \sum_{j=0}^{k} \binom{k}{j} (-1)^j \sum_{t=0}^{k-1} \binom{k-1}{t} (-j/\alpha)^t.$$

Interchanging the order of summation gives and using the familiar identity[1]

$$\sum_{j=0}^{k} \binom{k}{j} (-1)^j j^t \;=\; 0 \quad \text{for } 0 \le t < k,$$

we obtain $c_k(\alpha) = 0$.

One easily obtains (c) from (a) and (b) or by direct observation.

We now prove the continuity of $c_k(\alpha)$ for $k > 1$. First, note that

$$\text{for } k > 1, \text{ we may change the range of summation in (5.1) to } 0 \le j \le \alpha \tag{5.2}$$

because the $j = \alpha$ term is zero for $k > 1$. Since each term of the sum in (5.1) is continuous, the only possible discontinuities are at the positive integers where the number of summands in (5.1) changes. Using (5.2) eliminates this problem.

We now prove (e) by induction on $k$. In some open interval $(j_0, j_0 + 1)$, where $j_0$ is an integer, we find that $c_{k+1}$ is differentiable, namely,

$$\begin{aligned}
\frac{d}{d\alpha}\, c_{k+1}(\alpha) &= \frac{d}{d\alpha} \sum_{j=0}^{j_0} \binom{k+1}{j} (-1)^j \left(1 - \frac{j}{\alpha}\right)^k \\
&= \frac{k}{\alpha^2} \sum_{j=0}^{j_0} \binom{k+1}{j} j (-1)^j \left(1 - \frac{j}{\alpha}\right)^{k-1} \\
&= \frac{k(k+1)}{\alpha^2} \sum_{j=1}^{j_0} \binom{k}{j-1} (-1)^j \left(1 - \frac{j}{\alpha}\right)^{k-1} \\
&= \frac{k(k+1)}{\alpha^2} \sum_{j=0}^{j_0-1} \binom{k}{j} (-1)^{j+1} \left(1 - \frac{j+1}{\alpha}\right)^{k-1}.
\end{aligned}$$

Using $\left(1 - \frac{j+1}{\alpha}\right)^{k-1} = \left(1 - \frac{1}{\alpha}\right)^{k-1} \left(1 - \frac{j}{\alpha-1}\right)^{k-1}$, this becomes

$$\begin{aligned}
\frac{d}{d\alpha}\, c_{k+1}(\alpha) &= -\frac{k(k+1)}{\alpha^2} \left(1 - \frac{1}{\alpha}\right)^{k-1} \sum_{j=0}^{j_0-1} \binom{k}{j} (-1)^j \left(1 - \frac{j}{\alpha-1}\right)^{k-1} \\
&= -\frac{k(k+1)}{\alpha^2} \left(1 - \frac{1}{\alpha}\right)^{k-1} c_k(\alpha - 1).
\end{aligned} \tag{5.3}$$

By (e), or by (c) if $k = 1$, we see that this is strictly negative if $1 < \alpha < k + 1$. Since $c_{k+1}(\alpha)$ is continuous, it is strictly decreasing for $1 < \alpha < k + 1$. By continuity and (a), positivity follows. □

**Remark 4** The delay differential equation (5.3), plus continuity, completely determine the function $c_{k+1}$. Functions whose defining summations somewhat resemble ours were introduced by Bernstein [1] to give a constructive proof of the Weirstrass approximation theorem. □

[1] One way to prove the identity is to compute $(x\, d/dx)^t (1 - x)^k$ at $x = 1$.

## 6 Asymptotics for $E(r,k,n)$

We recall that $E(r,k,n)$ is the number of $k$-subsets of $[n]$ whose elements sum to $r$.

**Lemma 3** *Fix $0 < a < b < \infty$ and define $\alpha = r/n$. Then*

$$E(r,k,n) \;=\; \frac{c_k(\alpha)+o(1)}{k!}\binom{r-1}{k-1},$$

*uniformly as $n \to \infty$ with $a \le \alpha \le b$, $k^\alpha = o(n^{1/12})$ and $k = O(n^{1/4})$.*

*Proof* If $k \le \alpha$, then $E(r,k,n) = 0$ and $c_k(\alpha) = 0$. If $k = 1$, then $E(r,1,n) \;=\; \chi(1 \le r \le n)$, which equals $c_1(r/n)$ and the lemma is true. Thus we assume from now on that $k > 1$ and $k > \alpha$.

Let $C_{r,k}(\mathbf{S})$ (respectively, $P_{r,k}(\mathbf{S})$) denote the number of compositions (respectively, partitions) of $r$ into exactly $k$ parts satisfying $\mathbf{S}$. Thus

$$E(r,k,n) \;=\; P_{r,k}(\le n \text{ and } \ne) \;=\; (1/k!)\, C_{r,k}(\le n \text{ and } \ne), \tag{6.1}$$

where "$\le n$ and $\ne$" indicates the parts do not exceed $n$ and are distinct.

We require the following formulas, where $\emptyset$ indicates that there are no conditions on the parts.

$$\sum_{L=-A}^{B}\binom{A+L}{j-1}\binom{B-L}{k-1} \;=\; \binom{A+B+1}{j+k-1}, \tag{6.2}$$

$$C_{r,k}(\emptyset) \;=\; \binom{r-1}{k-1}, \tag{6.3}$$

$$P_{r,k}(>n) \;=\; P_{r-kn,k}(\emptyset), \tag{6.4}$$

$$P_{r,k}(\emptyset) \;=\; (1/k!)C_{r+\binom{k}{2},k}(\ne), \tag{6.5}$$

$$\binom{r-1}{k-1} \;\ge\; C_{r,k}(\ne) \;\ge\; \binom{r-1}{k-1} - \binom{k}{2}\binom{r-2}{k-2}. \tag{6.6}$$

Equation (6.2) can be proved by counting the $(j+k-1)$-element subsets of $[A+B+1]$ according to the size $A+L+1$ of the $j$-th smallest element in the subset. Equation (6.3) is a fundamental result in enumeration. (See, for example, [2] or [7].) Equations (6.4) and (6.5) are simple. The left side of (6.6) follows from (6.3). To obtain the right side, we subtract off an upper bound for the number of compositions with equal parts. This is obtained by choosing two positions that have equal parts, choosing their size (say $t$), and summing the number of ways to fill in the remaining $k-2$ parts arbitrarily:

$$\begin{aligned} C_{r,k}(\text{some} =) \;&\le\; \binom{k}{2}\sum_{t\ge 1} C_{r-2t,k-2}(\emptyset) \;=\; \binom{k}{2}\sum_{t\ge 1}\binom{r-2t-1}{k-3} \\ &\le\; \binom{k}{2}\sum_{t\ge 1}\binom{r-t-2}{k-3} \;=\; \binom{k}{2}\binom{r-2}{k-2}, \end{aligned} \tag{6.7}$$

where the last sum is (6.2) with $A=-1$, $B=r-2$, $j=1$, and $k$ replaced by $k-2$.

We return to the proof of the lemma. It follows from (6.7) that

$$C_{r,k}(\leq n \text{ and some } =) \;\leq\; \binom{k}{2}\binom{r-2}{k-2} \;=\; \binom{r-1}{k-1}o(k^3/r).$$

Thus, the lemma is equivalent to showing that

$$C_{r,k}(\leq n) \;=\; (c_k(\alpha)+o(1))\binom{r-1}{k-1} \tag{6.8}$$

uniformly under the constraints of the lemma plus $k>\alpha$ and $k>1$.

The number of compositions of $r$ with $k$ parts, in which $j$ parts larger than $n$ are distinguished, is clearly

$$(k)_j \sum_L P_{L,j}(>n)\, C_{r-L,k-j}(\emptyset),$$

where $(k)_j$ is the falling factorial. Hence, by inclusion/exclusion,

$$C_{r,k}(\leq n) \;=\sum_{0\leq j<\alpha}(-1)^j(k)_j \sum_L P_{L,j}(>n)\, C_{r-L,k-j}(\emptyset), \tag{6.9}$$

where the range of $j$ was obtained by noting that terms with $j>r/(n+1)$ vanish since $P_{L,j}(>n)=0$. We note that, since $\alpha\leq b$, $j$ is bounded.

The first step is to estimate the inner sum in (6.9), which we denote $\sigma(j,k,n,r)$:

$$\sigma(j,k,n,r) \;=\; \sum_L P_{L,j}(>n)\, C_{r-L,k-j}(\emptyset).$$

We claim that, for $0\leq j\leq k$,

$$\frac{1}{j!}\left[\binom{r-M}{k-1} \;-\; \binom{j}{2}\binom{r-M-1}{k-2}\right] \;\leq\; \sigma(j,k,n,r) \;\leq\; \frac{1}{j!}\binom{r-M}{k-1}, \tag{6.10}$$

where

$$M = jn - \binom{j}{2} + 1.$$

To see this, use (6.3)–(6.6) to obtain

$$\begin{aligned}
&\frac{1}{j!}\left[\binom{L-M}{j-1} \;-\; \binom{j}{2}\binom{L-M-1}{j-2}\right]\binom{r-L-1}{k-j-1}\\
&\quad\leq\; P_{L,j}(>n)\, C_{r-L,k-j}(\emptyset)\\
&\quad\leq\; \frac{1}{j!}\binom{L-M}{j-1}\binom{r-L-1}{k-j-1}
\end{aligned}$$

and then use (6.2) to sum over $L$.

First suppose that $j \geq \alpha - n^{-1/3}$. Since $j < \alpha$, there is at most one such $j$. In this case,

$$r - M \;\leq\; r - (\alpha - n^{-1/3})n + O(1) \;=\; n^{2/3} + O(1)$$

and so

$$\binom{r-M}{k-1} \;\leq\; \binom{r-1}{k-1}\left(\frac{r-M}{r-1}\right)^{k-1} \;\leq\; \binom{r-1}{k-1}\bigl(O(n^{-1/3})\bigr)^{k-1} \;\leq\; \binom{r-1}{k-1}O(n^{-(k-1)/4}).$$

Thus

$$\sigma(j,k,n,r) \;\leq\; \frac{1}{j!}\binom{r-1}{k-1}\,O(n^{-(k-1)/4}). \tag{6.11}$$

Now suppose that $j < \alpha - n^{-1/3}$. We have $r - M \;=\; \Omega(n^{2/3})$ and so

$$\begin{aligned}\binom{r-M}{k-1} \;&=\; \binom{r-1}{k-1}\left(\frac{r-M}{r-1}\right)^{k-1}\Bigl(1 + O(k^2/n^{2/3})\Bigr)\\ &=\; \binom{r-1}{k-1}\left(\frac{r-M}{r-1}\right)^{k-1}\Bigl(1 + O(n^{-1/6})\Bigr),\end{aligned}$$

where we have used

$$1 \;\geq\; \frac{(T)_{k-1}}{T^{k-1}} \;\geq\; (1-k/T)^k \;=\; 1 + O(k^2/T), \quad \text{provided } \; k^2/T = O(1). \tag{6.12}$$

Now, since $j \;<\; \alpha \;\leq\; b$, $r \;=\; \alpha n \;\geq\; an$, and $1 - j/\alpha \;>\; n^{-1/3}b$,

$$\frac{r-M}{r-1} \;=\; 1 - \frac{M}{r} + O(1/r) \;=\; 1 - \frac{j}{\alpha} + O(j^2/r).$$

Thus

$$\left(\frac{r-M}{r-1}\right)^{k-1} \;=\; \left(1 - \frac{j}{\alpha}\right)^{k-1}\Bigl(1 + O(kn^{-1/3})\Bigr)$$

and so, since $k = O(n^{1/4})$,

$$\binom{r-M}{k-1} \;=\; \binom{r-1}{k-1}\left(1 - \frac{j}{\alpha}\right)^{k-1}\Bigl(1 + O(n^{-1/12})\Bigr). \tag{6.13}$$

Further,

$$\binom{j}{2}\binom{r-M-1}{k-2} \;=\; \binom{r-M}{k-1}O(j^2k/r) \;=\; \binom{r-M}{k-1}O(n^{-1/4}) \tag{6.14}$$

and so, from (6.10), (6.13) and (6.14),

$$\sigma(j,k,n,r) \;=\; \frac{1}{j!}\binom{r-1}{k-1}\left(1 - \frac{j}{\alpha}\right)^{k-1}\Bigl(1 + O(n^{-1/12})\Bigr). \tag{6.15}$$

Substituting (6.15) into (6.9), we obtain

$$\begin{aligned} C_{r,k}(\le n) &= \sum_{0\le j<\alpha} (-1)^j \frac{(k)_j}{j!}\left(1-\frac{j}{\alpha}\right)^{k-1} \binom{r-1}{k-1} \\ &+ \sum_{0\le j<\alpha} (k)_j \binom{r-1}{k-1} O(n^{-1/12}) + T, \end{aligned} \tag{6.16}$$

where $T = 0$ if the fractional part of $\alpha$ exceeds $n^{-1/3}$. Otherwise, from (6.11) with $j=\beta=[\alpha]$,

$$|T| = \frac{(k)_\beta}{\beta!}\Big((1-\beta/\alpha)^{k-1} + O(n^{-(k-1)/4})\Big) = \frac{(k)_\beta}{\beta!}\,O(n^{-(k-1)/4}),$$

since $(1-\beta/\alpha) \le n^{-1/3}$. Thus $|T| = o(k^\alpha n^{-1/4})$ and so (6.16) becomes

$$C_{r,k}(\le n) = \Big(c_k(\alpha) + O(k^\alpha n^{-1/12})\Big)\binom{r-1}{k-1},$$

completing the proof. □

# 7 Proof of Theorem 1

This section is devoted to the proof of Theorem 1.

Our starting point is (4.2) and (4.4):

$$f(n,r) = D_{n-1}\left(\chi(0<\alpha\le 1) + \sum_{k>1} E(r,k,n)\frac{D_{n-k}}{D_{n-1}}\right)$$

$$\frac{D_{n-k}}{D_{n-1}} \sim (\lambda/Cn)^{k-1}.$$

The latter holds for each fixed $k$ and so holds uniformly for $k$ not exceeding some sufficiently slowly growing unbounded function of $n$, say $k \le k(n)$. We also insist that $k(n)$ be small enough that the conditions in Lemma 3 are satisfied when $k \le k(n)$. Now

$$\sum_{k>1} E(r,k,n)\frac{D_{n-k}}{D_{n-1}} \sim \sum_{k=2}^{k(n)} \frac{c_k(\alpha)+o(1)}{k!}\binom{r-1}{k-1}(\lambda/Cn)^{k-1} + \sum_{k>k(n)} E(r,k,n)\frac{D_{n-k}}{D_{n-1}}$$

uniformly for $a \le \alpha \le b$ as $n\to\infty$.

Consider the first sum on the right. Using (6.12) we have

$$\binom{r-1}{k-1} \sim \frac{(\alpha n)^{k-1}}{(k-1)!}$$

uniformly for $0 < k \le k(n)$ and so

$$\begin{aligned}
\sum_{k=2}^{k(n)} \frac{c_k(\alpha)+o(1)}{k!}\binom{r-1}{k-1}\left(\frac{\lambda}{Cn}\right)^{k-1} &\sim \sum_{k=2}^{k(n)} \frac{c_k(\alpha)+o(1)}{(k-1)!\,k!}\left(\frac{\alpha\lambda}{C}\right)^{k-1} \\
&\sim \sum_{k=2}^{k(n)} \frac{c_k(\alpha)+o(1)}{(k-1)!\,k!}\left(\frac{\alpha\lambda}{C}\right)^{k-1} + o(1)\sum_{k=2}^{k(n)} \frac{(\alpha\lambda/C)^{k-1}}{(k-1)!\,k!} \\
&\sim K_{\lambda/C}(\alpha) + o(1).
\end{aligned}$$

To complete the proof of the theorem, we must show that

$$\sum_{k>k(n)} E(r,k,n)\frac{D_{n-k}}{D_{n-1}} = o(1). \tag{7.1}$$

Using definitions and (6.6), we have

$$\begin{aligned}
E(r,k,n) &= P_{r,k}(\le n \text{ and } \ne) \le P_{r,k}(\ne) \\
&= \frac{1}{k!}C_{r,k}(\ne) \le \frac{1}{k!}\binom{r-1}{k-1} < \frac{(\alpha n)^{k-1}}{k!\,(k-1)!}.
\end{aligned} \tag{7.2}$$

From (4.3) with $t = 1$, we have $D_{n-1} \sim \lambda e^{-\lambda} G_n/n$. Combining this with $D_m \le G_m$, we see that there is some $B$ such that, for all sufficiently large $n$ and all $k < n$,

$$\frac{D_{n-k}}{D_{n-1}} \le B\frac{G_{n-k}}{G_n/n} = B\frac{n}{(n)_k}\frac{G_{n-k}/(n-k)!}{G_n/n!}.$$

By Stirling's formula and some crude estimates,

$$(n)_k = \frac{n!}{(n-k)!} > \frac{Bn^{1/2}(n/e)^n}{(n-k)^{1/2}((n-k)/e))^{n-k}} > B(n/e)^k$$

for some $B > 0$. Combining the two previous equations with Definition 2(b), we see that there is some $B$ such that, for all sufficiently large $n$, $D_{n-k}/D_{n-1} < B(B/n)^{k-1}$. Combining this with (7.2), the $k$-th term of (7.1) is bounded by

$$\frac{(\alpha n)^{k-1}}{k!\,(k-1)!}\,B(B/n)^{k-1} = \frac{B(\alpha B)^{k-1}}{k!\,(k-1)!}.$$

Hence (7.1) is the tail of a convergent series and so is $o(1)$ as $k(n) \to \infty$. □

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