0–1 Laws for Maps

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ABSTRACT

A class of finite structures has a 0–1 law with respect to a logic if every property expressible in the logic has a probability approaching a limit of 0 or 1 as the structure size grows. To formulate 0–1 laws for maps (*i.e.*, embeddings of graphs in a surface), it is necessary to represent maps as logical structures. Three such representations are given, the most general being the full cross representation based on Tutte's theory of combinatorial maps. The main result says that if a class of maps has two properties, richness and large representativity, then the corresponding class of full cross representations has a 0–1 law with respect to first-order logic. As a corollary the following classes of maps on a surface of fixed type have a first-order 0–1 law: all maps, smooth maps, 2-connected maps, 3-connected maps, triangular maps, 2-connected triangular maps, and 3-connected triangular maps.

Keywords: 0–1 law, maps

1. INTRODUCTION.

In probability theory a 0–1 law is a result that says all events of a certain kind have probability 0 or 1. For example, the Kolmogorov 0–1 law says all tail events have probability 0 or 1 [28]. In the study of random structures there are analogous 0–1 laws which say that in certain classes of structures, all properties expressible in some logic have an asymptotic probability of 0 or 1. The first significant result of this kind was due to Glebskiĭ *et al.* [23] and Fagin [18]. They showed that the

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probability that a sentence of first-order logic holds in a random finite structure over a relational vocabulary approaches either 0 or 1 as the size of the structure increases. For example, structures over a vocabulary with just a binary relation symbol E are directed graphs (possibly with loops). In first-order logic we can express the property that for any two distinct vertices there is a vertex with directed edges to both. This property must hold in almost all large directed graphs or almost none; an easy computation shows that it holds in almost all.

For some classes of structures, the 0–1 law may not hold for first-order logic. It is not difficult to find restricted classes where probabilities even fail to converge. Nonetheless, in many cases a 0–1 law does hold. 0–1 laws, and the techniques used to prove them have applications in areas such as expressiveness of logics in finite structures [9], expected time analyses of database query optimization [1], and approximation of NP-optimization and NP-counting problems [2, 13]. See Compton [12], Winkler [35], and Spencer [30] for surveys on 0–1 laws.

In some recent papers on 0–1 laws the combinatorial estimates have been considerably more difficult than the ones in the proof of the original 0–1 law. In this paper we will use recent work on some very difficult combinatorial problems, *viz.* the asymptotic enumeration of planar maps and, more generally, of maps on a surface. One of the difficulties we will face is simply formulating the notion of a 0–1 law for maps. A map is not a structure in the logical sense; it is a topological object. More precisely, a *map* is a connected graph (possibly with loops and multiple edges) embedded in a surface (or closed 2-manifold) such that all components of its complement are simply connected regions (or *faces*). Two maps are *isomorphic* if there is a homeomorphism between their surfaces taking vertices to vertices, edges to edges, and faces to faces. We enumerate maps by the number of edges where isomorphic maps are identified.

In formulating a 0–1 law for maps, it would be tempting to use the graph representation — a set together with an edge relation — as a representation of a map. However, this would pose problems in representing multiple edges. A more fundamental difficulty is that a graph contains no information about the embedding into the surface. We will require that the structures we use to represent maps must contain at least this much information. Fortunately, Tutte [32], extending earlier work of Edmonds [16], gave an elegant representation of maps as structures which meets this requirement. We will describe this representation in Section 3. These will be the structures used for our results.

We will make use of a well known logical tool, the Ehrenfeucht-Fraïssé game, to obtain our results. The proofs of our 0–1 laws are similar in some respects to the proofs of two well known theorems proved by means of Ehrenfeucht-Fraïssé games: Hanf's Theorem and Gaifman's Theorem (see Ebbinghaus and Flum [15]). Both of these theorems say that if two structures have the same kinds of "local" substructures, then they satisfy the same first-order sentences of a given quantifier rank. (We will not need the precise statements of these theorems here; the interested reader is referred to [15].) In Section 4, we describe two properties of classes of maps, *richness* and *large representativity*. Richness says that for a given planar map M in the class, almost all maps in the class have many submaps isomorphic to M. Large representativity, as we will see, implies that the "local" maps in the class are almost surely planar. The combination of these results implies that almost all maps in these classes have the same local submaps. It would seem that the 0–1 law should follow from from Hanf's Theorem or Gaifman's Theorem. Unfortunately, the precise definition of "local" is not the same. Therefore, we must develop the machinery in Section 4 to apply Ehrenfeucht-Fraïssé games to classes of maps.

Our main result, Theorem 5.2, says that a rich class of maps with large representativity has a first-order 0–1 law. We cite results in the literature showing that the following classes of maps on a surface of fixed type are rich and have large representativity, therefore have a first-order 0–1 law:

- all maps;
- smooth maps;
- 2-connected maps;
- 3-connected maps;
- triangular maps;
- 2-connected triangular maps;
- 3-connected triangular maps.

Definitions of the terms used here are given in Section 3.

2. EHRENFEUCHT-FRAÏSSÉ GAMES

The method of Ehrenfeucht-Fraïssé games is one of the few techniques from model theory that works well for finite models. In this section we will give a brief description of these games. We will assume the reader is familiar with the basics of first-order logic and model theory as found, *e.g.*, in [11].

Definition. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures over a common vocabulary. We write $\mathfrak{A} \equiv_m \mathfrak{B}$ if $\mathfrak{A}$ and $\mathfrak{B}$ satisfy precisely the same first-order sentences of quantifier rank m.*

The following result is straightforward.

Proposition 2.1. *For structures over a finite vocabulary with just relation and constant symbols, $\equiv_m$ is an equivalence relation of finite index.*

The Ehrenfeucht-Fraïssé game is an m-round game between two players, called *Spoiler* and *Duplicator* (this terminology is due to Joel Spencer). The game is played on a pair of structures $\mathfrak{A}$ and $\mathfrak{B}$ whose vocabulary Σ contains just relation and constant symbols; let $c_1, \ldots, c_k$ be the constant symbols. In each of the rounds, numbered $1, 2, \ldots, m$, Spoiler chooses one of the two structures and picks an element from it. He is not constrained to pick an element from the same structure he picked from in the previous round. Duplicator responds by picking a element from the other structure. Let d_i^{A} be the element chosen from $\mathfrak{A}$ (either by Spoiler or Duplicator), and d_i^{B} be the element chosen from $\mathfrak{B}$, in round i. As the notation suggests, we may view this as adding a new constant symbol d_i to the vocabulary in round i, and having the players pick an interpretation for it in the two structures. Duplicator wins if $\langle \mathfrak{A}, d_1^{\mathrm{A}}, \ldots, d_m^{\mathrm{A}} \rangle$ and $\langle \mathfrak{B}, d_1^{\mathrm{B}}, \ldots, d_m^{\mathrm{B}} \rangle$ satisfy precisely the same atomic formulas over the vocabulary $\Sigma \cup \{d_1, \ldots, d_m\}$; otherwise, Spoiler wins.

Another way of saying this is that Duplicator wins if $\{(c_i^{\mathfrak{A}}, c_i^{\mathfrak{B}}) \mid 1 \leq i \leq k\} \cup \{(d_i^{\mathfrak{A}}, d_i^{\mathfrak{B}}) \mid 1 \leq i \leq m\}$ is an isomorphism between substructures of $\mathfrak{A}$ and $\mathfrak{B}$.

The basic result concerning Ehrenfeucht-Fraïssé games (due to Fraïssé [19] and Ehrenfeucht [17]) is the following.

Theorem 2.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures over a vocabulary containing only relation and constant symbols. Duplicator has a winning strategy in the m-round Ehrenfeucht-Fraïssé game played on $\mathfrak{A}$ and $\mathfrak{B}$ if and only $\mathfrak{A} \equiv_m \mathfrak{B}$.*

Our discussion so far has concentrated on structures where there is just one sort. Many structures in mathematics contain more than one kind of object. Vector spaces contain scalars and vectors, for example. There is a standard way to model structures with several sorts of objects.

Definition. *Let S be a set of primitive elements called* sorts. *A* vocabulary Σ *consists of collection of constant and relation symbols, a mapping from the set of constant symbols to S (assigning the sort of each constant symbol), and a mapping from the set of relation symbols to sequences from S (giving the* arity *of each relation symbol). Usually Σ is allowed to contain function symbols as well, but in discussions of Ehrenfeucht-Fraïssé games they are excluded.*

A multi-sorted structure $\mathfrak{A}$ *over Σ consists of the following.*

(i) A collection of disjoint sets, or universes, *A_s, where s ranges over S.*
(ii) An element $c^{\mathfrak{A}} \in A_s$ for each constant symbol c of sort s.
(iii) A relations $R^{\mathfrak{A}} \subseteq A_{s_1} \times \cdots \times A_{s_k}$ for each relation symbol R of arity $(s_1, \ldots, s_k)$.

The first-order logic of multi-sorted structures is similar to classical first-order logic of one-sorted structures. Each sort has a designated set of variables. If R is a relation symbol of arity $(s_1, \ldots, s_k)$, and $x_1, \ldots, x_k$ are variables or constant symbols whose sorts are $s_1, \ldots, s_k$, respectively, then $R(x_1, \ldots, x_k)$ is an atomic formula. If x_1 and x_2 are variables or constant symbols of the same sort, then $x_1 = x_2$ is an atomic formula. We build more complex formulas from atomic formulas as in classical first-order logic, using Boolean operations and universal and existential quantification. Sentences are formulas without free variables. Truth values of formulas and sentences are defined in the usual way.

The rules for Ehrenfeucht-Fraïssé games on multi-sorted structures require only the added condition that Duplicator must respond to Spoiler's choice of an element in one structure with an element of the same sort in the other structure. It is easy to verify that Theorem 2.2 holds for multi-sorted structures.

We will conclude this section with an illustration of how Ehrenfeucht-Fraïssé games can be used to prove a 0–1 law. This example is due to Lynch [27], but our proof will differ in some places. This proof will serve as a model for the proof of the 0–1 law for maps in Theorem 5.2.

Let $\mathcal{C}$ be a class of (one-sorted) structures over the vocabulary consisting of a binary relation symbol E and a unary relation symbol U, where E always interprets a cycle. We may assume that a structure $\mathfrak{A}$ of cardinality n has a fixed universe $n = \{0, 1, \ldots, n-1\}$, and that $E^{\mathfrak{A}}$ is the set of pairs $(i, i+1)$, for $0 \leq i < n-1$, and

$(n-1, 0)$. We can identify these structures with circular words over the alphabet $\{0,1\}$. Figure 1 shows a circular word of size 8. The edges of E^{A} are indicated by arrows, and the elements of U^{A} are indicated by filled circles.

Let $\mathcal{C}_n$ be the class of structures in $\mathcal{C}$ with universe n. There are two natural probability measures on $\mathcal{C}_n$. The first is the uniform measure on the space of structures $\mathfrak{A} = \langle n, E^{\mathrm{A}}, U^{\mathrm{A}} \rangle$. There are 2^n structures in the space, one for each subset of n. This is the so-called *labeled* probability measure. The second is the uniform measure on the space of isomorphism types in $\mathcal{C}_n$. (An isomorphism type is the class of all structures isomorphic to a given structure.) This is the *unlabeled* probability measure on $\mathcal{C}$. These measures are clearly different. The unlabeled measure is closer in spirit to the measure we will use for maps; there we identify isomorphic maps rather than isomorphic structures. However, with regard to 0–1 laws on circular words, it will make no difference which of these two probability measures we take as the total variation distance between them is exponentially small and thus they behave the same way with respect to limit probabilities. Let μ_n denote one of the two measures on $\mathcal{C}_n$.

Definition. *For any sentence φ, let*

$$\mu_n(\varphi) = \mu_n(\{\mathfrak{A} \in \mathcal{C}_n \mid \mathfrak{A} \models \varphi\}).$$

Define $\mu(\varphi) = \lim_{n\to\infty} \mu_n(\varphi)$, whenever this limit exists. $\mu(\varphi)$ is the asymptotic probability *of φ. The first-order 0–1 law says that all first-order sentences have an asymptotic probability of 0 or 1. A property with asymptotic probability 1 is said to be* almost certain.

Theorem 2.3. *The class of circular words over $\{0,1\}$ has a first-order 0–1 law.*

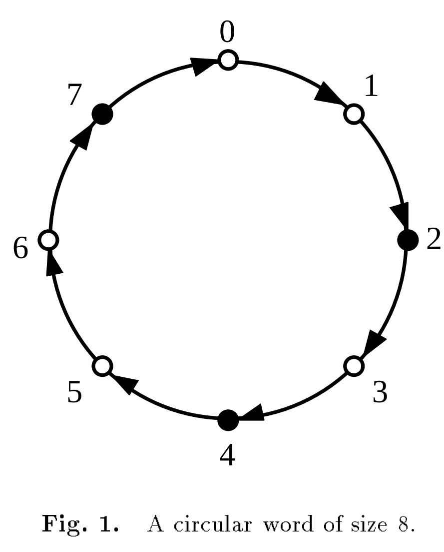

Fig. 1. A circular word of size 8.

Proof. Let $\mathfrak{A} \in \mathcal{C}_n$. Define a distance function on the universe of $\mathfrak{A}$: $d(a,b) = \min(|a-b|, n-|a-b|)$. That is, $d(a,b)$ is the minimal number of edges in E^{A} between a and b. For any nonnegative integer r and element a, $B^{\mathrm{A}}(a;r)$, the ball with center a and radius r, is the substructure whose universe is $\{x \mid d(a,x) \leq r\}$.

For all structures $\mathfrak{B} = \langle k, E^{\mathrm{B}}, U^{\mathrm{B}} \rangle$, where $k = \{0, 1, \ldots, k-1\}$ and E^{B} is the usual successor relation on k (*not* the circular successor relation), there is a constant $c > 0$ such that the asymptotic probability that a structure in $\mathcal{C}_n$ has at least cn substructures isomorphic to $\mathfrak{B}$ is 1. In other words, on circular words over the alphabet $\{0,1\}$, the likelihood of having many occurrences of a fixed (non-circular) subword is almost certain. This is easy to show for the labeled probability measure, and not difficult to show for the unlabeled probability measure.

As a consequence, for a fixed nonnegative integer j and a fixed structure $\mathfrak{B}$ as above, it is almost certain that for each sequence of elements $a_1, a_2, \ldots, a_j$ there is a substructure isomorphic to $\mathfrak{B}$ not containing any of the elements $a_1, a_2, \ldots, a_j$. The reason is that there can be only a bounded number of copies of $\mathfrak{B}$ containing some element $a_1, a_2, \ldots, a_j$, while the total number of copies of $\mathfrak{B}$ grows with n.

We will prove a 0–1 law for $\mathcal{C}$ by showing that for each m, there is an $\equiv_m$-class $\mathcal{E}$ with asymptotic probability 1. Thus, if φ is a sentence of quantifier rank m, the set of finite models of φ either contains $\mathcal{E}$, in which case $\mu(\varphi) = 1$, or is disjoint from $\mathcal{E}$, in which case $\mu(\varphi) = 0$.

Let $\mathcal{E}'$ be the class of structures such that for each $\mathfrak{B} = \langle k, E^{\mathrm{B}}, U^{\mathrm{B}} \rangle$ as above, where $k \leq 2^m + 1$, and each sequence of elements $a_1, a_2, \ldots, a_{m-1}$, there is a substructure isomorphic to $\mathfrak{B}$ not containing $a_1, a_2, \ldots, a_{m-1}$. Since m is fixed, by the remarks above, $\mathcal{E}'$ has asymptotic probability 1. We will show that Duplicator has a winning strategy in the m-round Ehrenfeucht-Fraïssé game played on each pair of structures from $\mathcal{E}'$. It follows that all of the structures in $\mathcal{E}'$ are equivalent with respect to $\equiv_m$ and, hence, there is a $\equiv_m$-class $\mathcal{E}$ containing $\mathcal{E}'$ with asymptotic probability 1.

Take $\mathfrak{A}, \mathfrak{B} \in \mathcal{E}'$. We present a winning strategy for Duplicator. Suppose that at the beginning of the k-th round of the game, the previously picked elements from $\mathfrak{A}$ are $a_1, a_2, \ldots, a_{k-1}$, and the previously picked elements from $\mathfrak{B}$ are $b_1, b_2, \ldots, b_{k-1}$. Suppose that Spoiler now picks an element from one of the structures, say a_k from $\mathfrak{A}$ (if he picks b_k from $\mathfrak{B}$, the strategy is symmetrical).

There are two cases.

If for all $i < k$, $d(a_i, a_k) > 2^{m-k}$, Duplicator chooses b_k in $\mathfrak{B}$ so that $B^{\mathrm{B}}(b_k; 2^{m-k})$ is isomorphic to $B^{\mathrm{A}}(a_k; 2^{m-k})$ and does not contain $b_1, b_2, \ldots, b_{k-1}$. Notice that the size of this substructure is at most $2^m + 1$ (the maximum being attained when $k = 1$) so it is always possible to find such a substructure in $\mathfrak{B}$.

If, on the other hand, for some $i < k$, $d(a_i, a_k) \leq 2^{m-k}$, Duplicator responds by choosing b_k from $\mathfrak{B}$ so that $d(a_i, a_k) = d(b_i, b_k)$ and the shortest path from a_i to a_k is in the same direction as the shortest path from b_i to b_k.

We must first show that this strategy is well defined. That is, if $d(a_i, a_k) \leq 2^{m-k}$ and $d(a_j, a_k) \leq 2^{m-k}$ for $i < j < k$, it should be possible to find a b_k in $\mathfrak{B}$ so that $d(a_i, a_k) = d(b_i, b_k)$, $d(a_j, a_k) = d(b_j, b_k)$, and a_k has the same orientation to a_i and a_j as b_k has to b_j and b_k.

Let us assume that Duplicator has been able to follow the strategy in rounds 1 through $k-1$. Construct a digraph $\mathcal{D}$ on the vertex set $\{1, 2, \ldots, m\}$ as the game progresses. Whenever $i < j$ and Spoiler chooses a_j so that $d(a_i, a_j) \leq 2^{m-j}$ or b_j so that $d(b_i, b_j) \leq 2^{m-j}$, directed edge (j, i) is added to $\mathcal{D}$. Several such edges may be added in a given round. We will see shortly that Duplicator's strategy will ensure that $d(a_i, a_j) \leq 2^{m-j}$ if and only if $d(b_i, b_j) \leq 2^{m-j}$.

If there is a path from j to i in $\mathcal{D}$ and i has out-degree 0, we say that i is an *anchor* for j. It is easy to see by induction that every element has a unique anchor. For example, if k has edges to i and j, where $i < j$, we know that i and j have unique anchors. Also,

$$d(a_i, a_j) \leq d(a_i, a_k) + d(a_k, a_j) \leq 2 \cdot 2^{m-k} \leq 2^{m-j}$$

so (j, i) is an edge of $\mathcal{D}$ and hence i and j have the same anchor.

We can now reformulate Duplicator's strategy as follows. When Spoiler picks a_k, find the anchor i for k. If $i = k$ then choose b_k so that $B^{\mathrm{B}}(b_k; 2^{m-k})$ is isomorphic to $B^{\mathrm{A}}(a_k; 2^{m-k})$ and does not contain $b_1, b_2, \ldots, b_{k-1}$. If $i \neq k$, then elements a_i and b_i were chosen so that $B^{\mathrm{A}}(a_i; 2^{m-i})$ is isomorphic to $B^{\mathrm{B}}(b_i; 2^{m-i})$. Since there is a path from k to i in $\mathcal{D}$, we have,

$$d(a_i, a_k) \leq 2^{m-k} + 2^{m-k-1} + \ldots + 2^{m-i-1} < 2^{m-i}.$$

We conclude that a_k is in $B^{\mathrm{A}}(a_i; 2^{m-i})$. In fact, for every $j < k$ with i as an anchor, a_j is in $B^{\mathrm{A}}(a_i; 2^{m-i})$, b_j is in $B^{\mathrm{B}}(b_i; 2^{m-i})$, and the isomorphism between $B^{\mathrm{A}}(a_i; 2^{m-i})$ and $B^{\mathrm{B}}(b_i; 2^{m-i})$ maps a_j to b_j. Duplicator simply continues this in round k: the isomorphism between $B^{\mathrm{A}}(a_i; 2^{m-i})$ and $B^{\mathrm{B}}(b_i; 2^{m-i})$ should map a_k to b_k. Notice that a_j may be played in $B^{\mathrm{A}}(a_i; 2^{m-i})$ in a way that does not make i an anchor for j. In this case Duplicator does not necessarily play the isomorphic image in $B^{\mathrm{B}}(a_i; 2^{m-i})$.

Clearly this strategy is well defined. Why is it a winning strategy for Duplicator? Elements related by E^{A} or E^{B} are distance 1 apart. The strategy ensures if $d(a_i, a_j) \leq 1$, then there is an isomorphic embedding of some ball containing a_i and a_j to $\mathfrak{B}$ so that a_i is mapped to b_i and a_j is mapped to b_j. The analogous statement holds when $d(b_i, b_j) \leq 1$. That is, a_i, a_j must satisfy the same relations as b_i, b_j. This is a winning position for Duplicator. ∎

3. MAPS AND THEIR REPRESENTATIONS

In this section we review basic notions concerning maps and discuss representations of maps as structures.

Definition. *A map M is a connected graph $\mathcal{G}$ embedded in a surface (or closed 2-manifold) $\mathcal{S}$ such that all components of $\mathcal{S} - \mathcal{G}$ are simply connected regions (or discs) called* faces. *This embedding must specify the images of both vertices and edges of the graph. Edges of M are mapped to homeomorphic copies of the open unit interval $(0,1)$. Edges together with their two end points are mapped to homeomorphic copies of the closed unit interval $[0,1]$. Images of distinct edges and vertices are disjoint.*

Two maps are isomorphic *if there is a homeomorphism from the surface of one to the surface of the other taking vertices to vertices, edges to edges, and faces to faces.*

Definition. *The* type *t of a map is given by the formula $2 - 2t = v - e + f$, where v, e, and f are the numbers of vertices, edges, and faces in the map. When the surface is* orientable *(*i.e., *has two sides), t is an integer giving the number of "holes" in the surface. (The converse is not true: there are non-orientable surfaces where t is an integer.) If $t = 0$ the surface is a sphere and the map is* planar*. When t is not an integer, it is of the form $s/2$, where s is a positive odd integer. A surface with s* cross-caps *has type $s/2$.*

Definition. *The* dual *$D(M)$ of a map M is a map on the same surface as M, with vertex set V and edge set E defined as follows. If x is a face of M, pick a point $d(x)$ in x; if y is an edge of M incident with faces x_1 and x_2 of M, let $d(y)$ be an edge of $D(M)$ connecting $d(x_1)$ and $d(x_2)$. Thus, $V = \{d(x) \mid x \text{ is a face of } M\}$ $E = \{d(y) \mid y \text{ is an edge of } M\}$. $D(D(M))$ is isomorphic to M. By the* degree *of a face x in M, we mean the degree of $d(x)$ in $D(M)$.*

Submaps will play an important role in our proofs.

Definition. *Let C be a cycle formed from some edges and their endpoints in a map. Suppose that cutting along C divides S into two pieces. Duplicate C so that each piece has a hole bounded by a copy of C and then fill each of these holes with discs. This gives two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ containing maps M_1 and M_2 which are* submaps *of M. Notice that the type of M is the sum of the types of M_1 and M_2. Whenever we speak of a submap, we will assume that the face formed by the added disc is distinguished, and, therefore, so too are the edges of the cycle C.*

There are several definitions of k-connectivity in the literature. We will use a variant of Tutte's definition [33]. The definition used here is from Graver and Watkins [24] and on graphs with at least three edges is equivalent to Tutte's definition.

Definition. *A graph $\mathcal{G}$ is k-connected if its girth (or shortest cycle size) is at least k and at least k vertices must be removed to separate the graph. Thus, 1-connectivity is the same as connectivity, 2-connectivity implies the absence of loops, and 3-connectivity implies the absence of multiple edges. A map is k-connected if the graph associated with it is k-connected.*

Definition. *A map is* smooth *if all its vertices have degree at least 2. A map is* triangular *if all its faces have degree 3. It is easy to see that faces of 2-connected triangular maps are true triangles (*i.e., *bounded by three distinct edges and three distinct vertices).*

To formulate 0–1 laws, we must find a way to represent a topological object, a map, as a logical structure. We cannot use just a set of vertices with an edge relation because this gives no information about how the map embeds in a surface, and, worse, does not allow multiple edges.

There are several possible ways to represent maps as multi-sorted structures, depending on how much information about the map we want to incorporate.

Definition. *In the* graph representation *of a map M there are three sorts v, e and f in Σ, corresponding to vertices, edges, and faces. That is, in the multi-sorted* *structure* $\mathfrak{A}$ *representing the map* M, A_v *is the set of vertices,* A_e *is the set of edges, and* A_f *is the set of faces.* Σ *contains two* incidence relation *symbols* I *and* J *whose arities are* (v, e) *and* (f, e), *respectively.* $I^{\mathfrak{A}}(x, z)$ *holds when vertex* x *and edge* z *are incident in* M, *and* $J^{\mathfrak{A}}(y, z)$ *holds when face* y *and edge* z *are incident in* M.

This representation allows multiple edges, but still does not give complete information about the embedding of M in the surface $\mathcal{S}$. For example, in Figure 2, there is no isomorphism from the first map to the second, but their graph representations are isomorphic. Each isomorphism of the graph representations maps v_i to w_i. (There is more than one isomorphism because the isomorphism is not uniquely determined on edges.) The maps are not isomorphic because the circular edge order about v_3 differs from the corresponding edge order about its isomorphic image w_3.

We would like to incorporate information about edge order in our structures. However, on an orientable surface there are two possible orientations and, thus, two ways to choose an edge order at vertices of degree 3 or more. On a non-orientable surface there is no consistent way to choose edge orders at each vertex. We might try to specify *adjacent* edges in the edge order at each vertex, but the maps in Figure 2 show that again nonisomorphic maps may have the same representation.

To address these problems, Edmonds [16], proposed a method of representing maps which was later used for a theory of combinatorial maps by Cori [14], Jacques [25], Walsh and Lehman [34], and Tutte [32]. In this representation a *dart* is an edge with an assigned direction. Thus, each edge is associated with two darts which we will say are *anti-darts* of one another. Notice that a dart is not quite the same as a directed edge (given as an ordered pair) since loops give rise to two darts. Observing the direction of a dart, we may speak of a dart being *out of* a vertex or *to* a vertex.

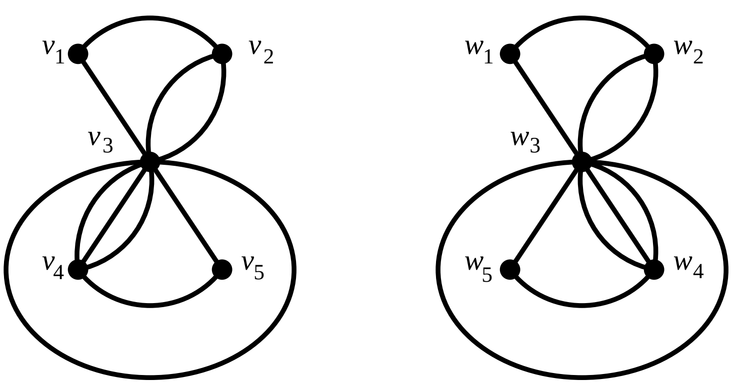

Fig. 2. Nonisomorphic maps with the same graph representation

The representation of Edmonds uses two permutations on the dart set A_d. The first permutation γ is an involution that maps each dart to its anti-dart. The second permutation ρ is determined by fixing an orientation of the surface and taking the permutation that maps each dart out of a vertex to the next dart in clockwise order out of the same vertex.

Represent a map on an orientable surface by the structure $\langle A_d, \gamma, \rho\rangle$. Why does this uniquely determine the map? Consider the permutation $\sigma = \rho\gamma$ (*i.e.*, the permutation formed by applying γ then ρ). Consider the dart sequences around each face in the map. Each dart occurs in two such face sequences, one clockwise and one counterclockwise. The permutation σ maps each dart to the next dart in its *counterclockwise* face sequence. Note that the cycles of γ correspond to the edges of the map, the cycles of ρ correspond to the vertices, and the cycles of σ correspond to faces. Incidence between a vertex and edge (or a face and an edge) occurs precisely when their corresponding cycles have an element in common. Thus, the graph structure of a map can be recovered from γ and ρ. The cycles of σ tell us how to "glue" discs onto face cycles to complete the map. This representation gives a nice interpretation of map duality: if $\langle A_d, \gamma, \rho\rangle$ represents a map, $\langle A_d, \gamma, \sigma\rangle$ represents its dual.

For our purposes, this representation is inadequate on two counts. First, it does not work for maps on a non-orientable surface. Second, on an orientable surface it does not give a unique representation: $\langle A_d, \gamma, \rho\rangle$ and $\langle A_d, \gamma, \rho^{-1}\rangle$ represent the same map.

Tutte [31, 33] extended Edmond's representation to overcome some of these difficulties. He used *crosses* rather than darts. A *cross* is an edge with a direction and designated side. A cross is determined by a dart in the map and a crossing dart the dual map. The dart in the dual map points to the designated side of the rooted edge. (The term *rooted edge* is often used in the literature rather than *cross*.)

We will say that two crosses on the same edge are *anti-crosses* of one another if they have the same designated side but opposite directions. We will say that two crosses on the same edge are *co-crosses* of one another if they have the same direction but opposite designated sides. For a fixed map M, let γ be the involution that maps each cross to its anti-cross and δ be the involution that maps each cross to its co-cross. Notice that γ and δ generate a group of order 4.

Figure 3 illustrates the definition of cross permutation ρ. There are three vertices v_1, v_2, v_3 from a map M and three vertices f_1, f_2, f_3 from $D(M)$ pictured. Edges

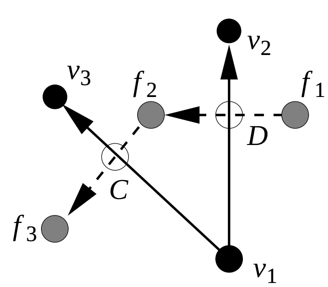

Fig. 3. Drawing used to define permutation ρ.

from M between vertices v_1 and v_2 and between vertices v_1 and v_3 have been given the directions indicated. The dual edges (from $D(M)$) between f_1 and f_2 and between f_2 and f_3 have been given the directions indicated. Thus, two crosses C and D are pictured. When a situation like this occurs we will define $\rho(C) = D$. To be precise, if a cross C is given by a directed edge e from v_1 to v_2 and directed dual edge from f_1 to f_2, then there exists a cross D given by a directed edge from v_1 to some v_3 and directed dual edge from f_2 to some f_3. Moreover, if we stipulate that C and D are not co-crosses unless v_1 has degree 1, this defines D uniquely. Therefore, ρ is well defined.

The subgroup generated by ρ and $\gamma\delta$ has precisely two orbits when the surface is orientable. If the surface is non-orientable, this subgroup has just one orbit (*i.e.*, it acts transitively on the set of crosses).

Let A_c be the set of all crosses in a map M. The structure $\langle A_c, \gamma, \delta, \rho\rangle$ represents M. The orbits of the subgroup generated by γ and δ correspond to edges of the map. As we have seen, each of these orbits consists of four crosses. The orbits of the subgroup generated by ρ and δ correspond to the vertices of M. Each of these orbits consists of cycles of ρ "paired" by δ. That is, if $\rho(x) = y$ then $\rho(\delta(y)) = \delta(x)$. Notice that δ is an isomorphism from $\langle A_c, \gamma, \delta, \rho\rangle$ to $\langle A_c, \gamma, \delta, \rho^{-1}\rangle$.

Crosses y and $\sigma(y) = \rho\gamma\delta(y)$ are always successive crosses in a cycle around some face x. Also, the designated sides of y and $\sigma(y)$ are toward face x. $\langle A_c, \gamma, \delta, \sigma\rangle$ represents the dual map of M. The cycles of σ are also paired by δ and determine how discs should be glued to edges to form the surface of M.

Let us modify this representation in three ways. First, so that we may use Ehrenfeucht-Fraïssé game techniques, we represent permutations as binary relations rather than unary functions. Second, so that we can quantify over faces and vertices, we use multi-sorted structures. Third, we add an equivalence relation that holds between crosses along the same edge. This last relation is not necessary, since it can be defined from the other relations, but it is convenient to have.

Definition. *In the* cross representation *of a map M there are three sorts v, c and f, corresponding to vertices, crosses, and faces of M. Σ contains two* incidence relation *symbols I and J whose arities are (v, c) and (f, c), respectively. $I^{\mathbf{A}}(x, z)$ holds when cross z is a cross* out *of vertex x. $J^{\mathbf{A}}(y, z)$ holds when y is the face on the designated side of cross z. Also, Σ contains binary relation symbols G, D, R, and E, each of arity (c, c). $G^{\mathbf{A}}(x, y)$ holds if $\gamma(x) = y$. $D^{\mathbf{A}}$ and $R^{\mathbf{A}}$ are defined similarly for δ and ρ. $E^{\mathbf{A}}(x, y)$ holds if x and y are crosses along the same edge.*

We will prove 0–1 laws both for graph representations and cross representations. The *size* of a structure is the number of *edges* its map contains (or a quarter the number of crosses).

There is one other representation we will use. It extends the cross representation in a nontrivial way, but the proof of the 0–1 law goes through with only minor modifications.

Definition. *The* full cross representation *vocabulary contains, in addition to the symbols in the cross representation vocabulary, a symbol T of arity (c, c, c, c). $T^{\mathbf{A}}(z_1, z_2, z_3, z_4)$ holds if for some i, $\rho^i(z_1) = z_2$ and $\rho^i(z_3) = z_4$, and for some j, $\rho^j(z_1) = z_3$. In other words, z_1, z_2, z_3, z_4 are crosses out of the same vertex, have* *the same orientation, and the "angle" between* z_1 *and* z_2 *is the same as the one between* z_3 *and* z_4*.*

The crucial fact about the full cross representation that enables us to prove a 0–1 law is that this added relation is local. There are many others we could have added.

Notice that the graph representation of a map is first-order interpretable in the cross representation, in the following sense. Let us write (with a slight abuse of notation) $\mathfrak{A}/E^{\mathbf{A}}$ to denote the graph representation formed from the cross representation by taking the quotient of A_c by $E^{\mathbf{A}}$ and interpreting I and J in the obvious way. There is a straightforward mapping taking each sentence φ over the graph representation vocabulary to a sentence φ' over the cross representation vocabulary so that $\mathfrak{A}/E^{\mathbf{A}} \models \varphi$ if and only if $\mathfrak{A} \models \varphi'$. Simply replace each occurrence of a subformula of the form $I(x, z)$ with a formula $\exists z'(I(x, z') \wedge E(z', z))$, each occurrence of a subformula of the form $J(x, z)$ with a formula $\exists z'(J(x, z') \wedge E(z', z))$, and each occurrence of a subformula of the form $z = z'$ where z and z' have sort e with a formula $E(z, z')$. This shows that if we prove a 0–1 law for cross representations (or full cross representations) we prove one for graph representations as well (assuming that the distribution is given by the uniform distribution on maps, *not* on graph representations).

4. DISTANCES ON MAPS

A proof of a 0–1 law for maps along the lines of the proof of the 0–1 law for circular words requires a notion of distance in maps. The distance function on circular words has a crucial property not shared by the usual distance function on the vertex set of a map: for large circular words $\mathfrak{A}$ and $\mathfrak{B}$, if a ball of radius r in $\mathfrak{A}$ is embedded in $\mathfrak{B}$, its image is also a ball of radius r in $\mathfrak{B}$. This was needed for Duplicator's strategy based on isomorphisms between balls. We must define a different notion of distance on maps. Also, this definition should make sense in our representations. We will confine most of our remarks to full cross representations, which are the most general, but everything we do in this section works for graph representations and cross representations as well.

We first present the notion of a quadrangulation of a map. Note that this is not the usual definition of quadrangulation first given by Brown [10].

Definition. *The* quadrangulation *of a map* M*, denoted* $Q(M)$*, can be viewed as the map formed by superimposing the dual* $D(M)$ *over* M*. More specifically,* $Q(M)$ *is a map on the same surface as* M*, with vertex set* V *and edge set* E *defined as follows. Let* A_v*,* A_e*,* A_f *be the sets of vertices, edges, and faces of* M*. If* x *is in* A_e *or* A_f*, pick a point* $q(x)$ *in* x*; if* x *is in* A_v*, let* $q(x) = x$*.* $V = \{q(x) \mid x \in A_v \cup A_e \cup A_f\}$*.* E *contains two kinds of edges. If vertex* x *and edge* z *are incident in* M*, then* E *contains an edge between* $q(x)$ *and* $q(z)$*. If face* y *and edge* z *are incident in* M*, then* E *contains an edge between* $q(y)$ *and* $q(z)$*.*

There is a subtle issue in this definition that will not matter for our applications, but may have troubled the observant reader. If vertex x is at both ends of edge y, then there should be two edges between $q(x)$ and $q(y)$. If face y is incident with both sides of edge z, then there should be two edges between $q(y)$ and $q(z)$.

Figure 4 illustrates the reason $Q(M)$ is called a quadrangulation. We view $Q(M)$ as being formed first by putting a vertex on each edge of M, thereby dividing each edge into two edges, then by adding a new vertex in the middle of each face and connecting it to the added vertices on the edges around the face. Faces of $Q(M)$ are quadrangles or degenerate quadrangles. $Q(M)$ is a bipartite graph: all edges connect a vertex $q(x)$ where $x \in A_v \cup A_f$ to a vertex $q(y)$ where $y \in A_e$.

We are now ready to give a definition of distance in graph representations and full cross representations.

Definition. *Let M be a map and $\mathfrak{A}$ be its graph representation. The universes of $\mathfrak{A}$ are A_v, A_e, and A_f, the vertex, edge, and face sets of M. For $x, y \in A_v \cup A_e \cup A_f$, define $d^{\mathfrak{A}}(x,y)$ to be the length of the shortest path from $q(x)$ to $q(y)$ in $Q(M)$. That is, we transfer the usual graph distance function on $Q(M)$ to $\mathfrak{A}$. We will write $d(x,y)$ rather than $d^{\mathfrak{A}}(x,y)$ when $\mathfrak{A}$ is clear from context. Clearly, d is a metric.*

If $\mathfrak{A}$ is the cross or full cross representation of M, the universes are A_v, A_c and A_f. If z_1, z_2, z_3, z_4 are the crosses on edge z, define $q(z_i) = q(z)$ for $i = 1, 2, 3, 4$. Now define the distance function d on $\mathfrak{A}$ in the same way as for the graph representations. Here d is not a metric, since the distance between a cross and its anti-cross or co-cross is 0, but d is a pseudo-metric; i.e., *$d(x,y) = d(y,x)$ and $d(x,z) \le d(x,y) + d(y,z)$.*

This distance function is important because of its relation to Ehrenfeucht-Fraïssé games played on map representations.

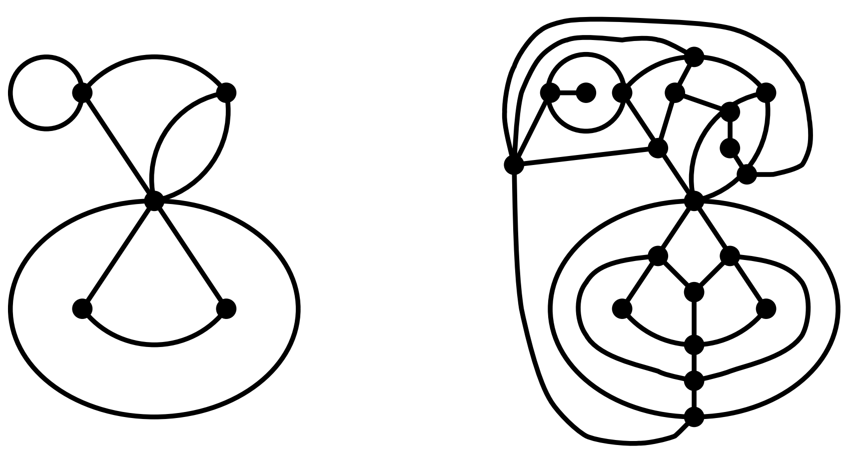

Fig. 4. A map M and its quadrangulation $Q(M)$

Proposition 4.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be full cross representations of maps. Augment the full cross representation vocabulary with with additional constant symbols c and c' interpreted by a, a' in $\mathfrak{A}$ and b, b' in $\mathfrak{B}$. Suppose $\langle \mathfrak{A}, a, a' \rangle \equiv_m \langle \mathfrak{B}, b, b' \rangle$, where $m \geq 1$. If $d^{\mathrm{A}}(a, a') \leq 2^{m-1}$, then $d^{\mathrm{A}}(a, a') = d^{\mathrm{B}}(b, b')$.*

Proof. For each $l \leq 2^{m-1}$ and each pair of sorts $s, t \in \{v, c, f\}$, there is a formula $\varphi_l^{s,t}(x, y)$ of quantifier rank at most m that holds precisely when $d(x, y) \leq l$. Here the sorts of x and y are s and t. We prove this by induction on m.

Consider first the base case $m = 1$. When $s = t = v$ or $s = t = f$, $\varphi_0^{s,t}(x, y)$ is the formula $x = y$; when $s = t = c$, it is the formula $E(x, y)$. In all other cases, $\varphi_0^{s,t}(x, y)$ is an invalid formula. It is easy to write a formula $\varphi_1^{s,t}(x, y)$ of quantifier rank 1 for the various cases of s and t. For example, if $s = v$ and $t = c$, then $\varphi_1^{s,t}(x, y)$ is $\exists z'(I(x, z') \wedge E(z', z))$.

When $m > 1$ we must specify formulas $\varphi_l^{s,t}(x, y)$ where $2^{m-2} < l \leq 2^{m-1}$; formulas for smaller l are given by the induction hypothesis. Since $l \leq 2^{m-1}$, $\lceil l/2 \rceil$ and $\lfloor l/2 \rfloor$ are at most 2^{m-2}, so there are formulas $\varphi_{\lceil l/2 \rceil}^{s,u}(x, z)$ and $\varphi_{\lfloor l/2 \rfloor}^{u,t}(z, y)$ of quantifier rank at most $m - 1$ for each sort u. Let $\varphi_l^{s,t}(x, y)$ be the disjunction of the formulas

$$\exists z(\varphi_{\lceil l/2 \rceil}^{s,u}(x, z) \wedge \varphi_{\lfloor l/2 \rfloor}^{u,t}(z, y))$$

where u ranges over the possible intermediate sorts. This formula is of quantifier rank at most m.

Now $\varphi_l^{s,t}(a, a')$ holds in the first structure if and only if $\varphi_l^{s,t}(b, b')$ holds in the second. Thus, if $d^{\mathrm{A}}(a, a') \leq 2^{m-1}$, then $d^{\mathrm{A}}(a, a') = d^{\mathrm{B}}(b, b')$. ∎

The following definitions will be needed later.

Definition. *The edge-width of a map M is the length (*i.e., *number of edges) of the smallest noncontractible curve in the graph of M. The* representativity *of a map M on a surface $\mathcal{S}$ is the smallest number of intersections a noncontractible curve in $\mathcal{S}$ has with the graph of M.*

The results below show that the representativity of M is closely related to the edge-width of $Q(M)$.

Lemma 4.2. *Let M be a map on a surface $\mathcal{S}$ and $\mathcal{R}$ be a closed subset of $\mathcal{S}$ composed of some of the vertices, edges and faces from M. Suppose that C is a closed curve in $\mathcal{R}$ and that C intersects the graph of M in precisely s points. Then C can be smoothly deformed into a curve C' in $\mathcal{R}$ made of vertices and edges in $Q(M)$ such that the number of vertices of $Q(M)$ on C' is at most $4s$.*

Proof. List the elements of $A_v \cup A_e \cup A_f$ on C in the (cyclic) order they occur on C. Every second element in this list is in A_f, so the length of the list is $2s$. We may write it $x_1, x_2, \ldots, x_{2s}$. Now expand the list in the following way. If one of the elements x_i or x_{i+1} is in A_v and the other is in A_f, there must be a $y \in A_e$ incident with both x_i and x_{i+1}. Since $\mathcal{R}$ is closed y is in $\mathcal{R}$. Insert y between x_i and x_{i+1} in the list. Do this also between x_{2s} and x_1 if one is in A_v and the other is in A_f. This gives a new list $y_1, y_2, \ldots, y_t$ where $t \leq 4s$; $q(y_1), q(y_2), \ldots, q(y_t)$ are consecutive vertices on a curve C' in $Q(M)$. Clearly, C can be smoothly deformed into C' and C' is contained in $\mathcal{R}$. ∎

Proposition 4.3. *If M has representativity s and $Q(M)$ has edge-width t, then $t/4 \leq s \leq t$.*

Proof. The inequality $t/4 \leq s$ follows from Lemma 4.2. For the other direction let A_v, A_e, and A_f be the sets of vertices, edges, and faces of M. Take a noncontractible curve C of length t in $Q(M)$. Recall that the vertices of $Q(M)$ are points of the form $q(x)$, where $x \in A_v \cup A_e \cup A_f$. Enumerate the points of this form on C in cyclic order: $q(x_1), q(x_2), \ldots, q(x_t)$. Whenever $x_i \in A_v \cup A_e$, $q(x_i)$ is on M. C may intersect M in other points, but by deforming C we form another noncontractible curve C' that intersects the graph of M only in the points $q(x_i)$ where $x_i \in A_v \cup A_e$. C' intersects the graph of M in at most t points, so $s \leq t$. ∎

Next we give the definition of a ball of radius r in one of our map representations. We have to work out some technical problems to get properties needed for the proof of a 0–1 law.

Definition. *Let $\mathfrak{A}$ be the graph representation, cross representation, or full cross representation of a map M. Suppose that $a \in A_e$ (or A_c in the case of the cross representations) and r is an even positive integer, or that $a \in A_v \cup A_f$ and r is an odd integer greater than 1. $B^{\mathrm{A}}(a; r)$ is the substructure of $\mathfrak{A}$ whose universes are as follows.*

(i) The set of elements $x \in A_f$ such that $d(a, x) < r$.
(ii) The set of elements $x \in A_e$ incident with an element in (i).
(iii) The set of elements $x \in A_v$ incident with an element in (ii).

We will refer to this as the ball of center a and radius r. *The region covered by faces, edges, and vertices in (i)-(iii) above is closed. It is easy to see that the points in (ii) and (iii) added to the points in (i) form the closure of points in (i).*

Let $S^{\mathrm{A}}(a; r)$ be the substructure of $\mathfrak{A}$ whose universes are as follows.

(i′) No elements of A_f.
(ii′) The set of elements $x \in A_e$ incident with some $y_1 \in B^{\mathrm{A}}(a; r)$ and some $y_2 \notin B^{\mathrm{A}}(a; r)$, where y_1 and y_2 are faces.
(iii′) The set of elements $x \in A_v$ incident with an element in (ii′).

The following results are immediate.

Proposition 4.4. $S^{\mathrm{A}}(a; r) \subseteq B^{\mathrm{A}}(a; r)$.

Proposition 4.5. *Every element of A_e in $S^{\mathrm{A}}(a; r)$ is distance r from a and is incident with some $y_1, y_2 \in A_f$ such that $d(a, y_1) = r - 1$ and $d(a, y_2) = r + 1$.*

Remark. Elements in A_v on $S^{\mathrm{A}}(a; r)$ may be distance $r - 1$ or $r + 1$ from a. There may be elements of A_e which are distance r from a but not in $B^{\mathrm{A}}(a; r)$. The union of edges and vertices in $S^{\mathrm{A}}(a; r)$ may not be a simple closed curve.

The next two propositions show that despite these shortcomings, $B^{\mathrm{A}}(a; r)$ behaves well enough to be of use in the proof of a 0–1 law.

Proposition 4.6. *Let M be a map on $\mathcal{S}$ and $B^{\mathfrak{A}}(a;r)$ be a ball of radius r in $\mathfrak{A}$, the full cross representation of M. Let M' be a map on $\mathcal{S}'$ with full cross representation $\mathfrak{A}'$. Suppose that h is a homeomorphism from the region $\mathcal{R}$ covered by $B^{\mathfrak{A}}(a;r)$ in $\mathcal{S}$ to a region $\mathcal{R}'$ in $\mathcal{S}'$ such that the images of the vertices, edges, and faces of M in $\mathcal{R}$ are, respectively, vertices, edges, and faces of M'. Then $\mathcal{R}'$ is precisely the region covered by $B^{\mathfrak{A}'}(h(a);r)$, and for all b in $B^{\mathfrak{A}}(a;r)$, $d^{\mathfrak{A}}(a,b) = d^{\mathfrak{A}'}(h(a),h(b))$.*

Proof. Every face y in $\mathcal{R}'$ is of the form $h(x)$, where x is a face in $\mathcal{R}$ with $d^{\mathfrak{A}}(a,x) < r$. Therefore, $d^{\mathfrak{A}'}(h(a),y) < r$ and y is in $B^{\mathfrak{A}'}(h(a);r)$. Every edge y_1 in $\mathcal{R}'$ is of the form $h(x_1)$ for some edge x_1 in $\mathcal{R}$. But x_1 must be incident with some face x_2 in $\mathcal{R}$, so y_1 is incident with a face $y_2 = h(x_2)$ in $B^{\mathfrak{A}'}(h(a);r)$. Since $B^{\mathfrak{A}'}(h(a);r)$ is closed, it contains y_1. Similarly, every vertex y_1 in $\mathcal{R}'$ is incident with an edge y_2 in $B^{\mathfrak{A}'}(h(a);r)$, so y_1 is in $B^{\mathfrak{A}'}(h(a);r)$. Thus, $\mathcal{R}' \subseteq B^{\mathfrak{A}'}(h(a);r)$.

We will derive a contradiction from the assumption that there is an element in $B^{\mathfrak{A}'}(h(a);r) - \mathcal{R}'$. Among such elements choose y' with minimal distance to $h(a)$. Thus, y' is incident with some $y \in \mathcal{R}'$ where $d^{\mathfrak{A}'}(h(a),y') = d^{\mathfrak{A}'}(h(a),y) + 1$.

Now y' is not a vertex, for then y would be an edge in $\mathcal{R}'$, and since $\mathcal{R}'$ is closed it would contain y'. By the same argument, it is not the case that y' is an edge and y is a face.

Suppose that y' is an edge and y is a vertex. Then $y = h(x)$ for some x in $\mathcal{R}$. Let $y_0, y_1, \ldots, y_k$ be a shortest path from $h(a) = y_0$ to $y = y_k$ (*i.e.*, $q(y_0), q(y_1), \ldots, q(y_k)$ are vertices along a shortest path from $q(h(a))$ to $q(y)$ in $Q(M')$). We may assume each y_i is in $\mathcal{R}'$, since they their distances to $h(a)$ are all less than that of y' and y' was chosen to have minimal distance to $h(a)$ among elements of $B^{\mathfrak{A}'}(h(a);r) - \mathcal{R}'$. For each y_i there is an $x_i \in \mathcal{R}$ such that $x_i = h(y_i)$. Furthermore, x_k is in $S^{\mathfrak{A}}(a;r)$, for otherwise x_k would be in the interior of $\mathcal{R}$, $y = y_k = h(x_k)$ would be in the interior of $\mathcal{R}'$, and y' would be in $\mathcal{R}'$. But $x_0, x_1, \ldots, x_k$ is a shortest path from $a = x_0$ to x_k so k is either $r-1$ or $r+1$. It cannot be $r+1$ because then $d^{\mathfrak{A}'}(h(a),y') = r+2$, which contradicts y' being an element of $B^{\mathfrak{A}'}(h(a);r)$. Thus, k is $r-1$ and $d^{\mathfrak{A}'}(h(a),y') = r$. Now y' is incident with a face z in $B^{\mathfrak{A}'}(h(a);r)$. The maximum distance of any face in $B^{\mathfrak{A}'}(h(a);r)$ to $h(a)$ is $r-1$, so again since y' has minimal distance to $h(a)$ among elements of $B^{\mathfrak{A}'}(h(a);r) - \mathcal{R}'$, z is in $\mathcal{R}'$. Since $\mathcal{R}'$ is closed, it contains y', which is a contradiction.

Finally, suppose that y' is a face and y is an edge. The argument here is similar to the one in the previous paragraph. We let $y_0, y_1, \ldots, y_k$ be a shortest path from $h(a) = y_0$ to $y = y_k$ (*i.e.*, $q(y_0), q(y_1), \ldots, q(y_k)$ are vertices along a shortest path from $q(h(a))$ to $q(y)$ in $Q(M')$) with each y_i in $\mathcal{R}'$). For each y_i there is an $x_i \in \mathcal{R}$ such that $x_i = h(y_i)$ and x_k is in $S^{\mathfrak{A}}(a;r)$. Now k is r, so $d^{\mathfrak{A}'}(h(a),y') = r+1$, which contradicts y' being an element of $B^{\mathfrak{A}'}(h(a);r)$.

The rest of the proposition follows easily. If b is in $B^{\mathfrak{A}}(a;r)$, there is a shortest path from a to b in $B^{\mathfrak{A}}(a;r)$, and, thus, its image under h is in $B^{\mathfrak{A}'}(h(a);r)$. Conversely, a shortest path from $h(a)$ to $h(b)$ is in $B^{\mathfrak{A}'}(h(a);r)$, and, thus, its pre-image under h is in $B^{\mathfrak{A}}(a;r)$. Therefore, $d^{\mathfrak{A}}(a,b) = d^{\mathfrak{A}'}(h(a),h(b))$. ∎

In the following proposition, when we say that $B^{\mathfrak{A}}(a;r)$ is planar, we mean that region covered by $B^{\mathfrak{A}}(a;r)$ is planar.

Proposition 4.7. *Suppose that a map M has representativity at least $2r+4$. Then $B^{\mathrm{A}}(a;r)$ is planar.*

Proof. We show that any closed curve C in $B^{\mathrm{A}}(a;r)$ is contractible in the surface of M (although not necessarily in $B^{\mathrm{A}}(a;r)$ since it may not be simply connected). By Lemma 4.2 C can be smoothly deformed into another curve C' in $B^{\mathrm{A}}(a;r)$, where C' is contained in $Q(M)$. List the vertices of C' in cyclic order: $x_1, x_2, \ldots, x_s$. Since these vertices are all in $B^{\mathrm{A}}(a;r)$, for each i there is a path p_i of length at most $r+1$ from x_i to a. Let p_i^* be the reversal of p_i. C can be smoothly deformed into a curve consisting of p_1, p_1^*, followed by (x_1, x_2), followed by p_2, p_2^*, and so on, to p_s, p_s^*, followed by (x_s, x_1). (See Figure 5.) This curve is a union of curves of the form p_i^*, followed by (x_i, x_{i+1}), followed by p_{i+1}, and p_s^*, followed by (x_s, x_1), followed by p_1. Each of these curves has length at most $2r+3$. By Proposition 4.3, $Q(M)$ has edge-width at least $2r+4$, so each of the curves can be contracted to the point a. Therefore, C can be contracted to a. We conclude that $B^{\mathrm{A}}(a;r)$ is planar. ∎

5. 0–1 LAWS FOR VARIOUS CLASSES OF MAPS

The main theorem of the paper, Theorem 5.2, gives 0–1 laws for various classes of maps. Throughout this section, we work with a fixed class of maps and μ_n will denote the uniform probability measure on the subclass of n-edge maps, where isomorphic maps are identified. We will assume that all maps in the class are on a surface of type t, where t is a fixed number of the form $s/2$ for some non-negative integer s. Let $\mathcal{C}$ be the class of full cross representations of maps in the class. Since the full cross representation of a map is unique, we may regard μ_n as the uniform probability measure on the class $\mathcal{C}_n$ of isomorphism types of full cross representations of n-edge maps in the class. (The same remark applies to

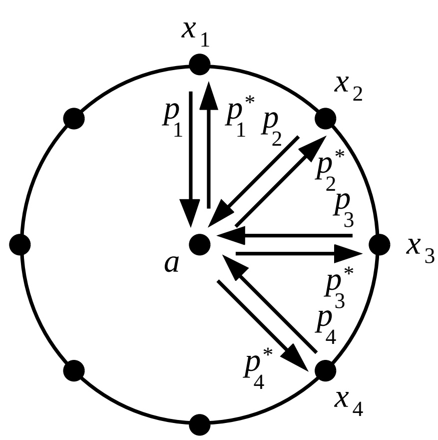

Fig. 5. Diagram used in the proof of Proposition 4.7.

cross representations, but *not* to graph representations. The probability of a graph representation is weighted proportionally to the number of embeddings it has in the surface.)

For any sentence φ, let

$$\mu_n(\varphi) = \mu_n(\{\mathfrak{A} \in \mathcal{C}_n \mid \mathfrak{A} \models \varphi\}).$$

Define $\mu(\varphi) = \lim_{n\to\infty} \mu_n(\varphi)$, whenever this limit exists. $\mu(\varphi)$ is the *asymptotic probability* of φ.

Two properties of classes of maps will be needed to formulate our main result.

Definition. *The class $\mathcal{C}$ is* rich *if for every planar map P (with a distinguished face) that occurs as a submap of some map in $\mathcal{C}$ and for every $k > 0$,*

$$\lim_{n\to\infty} \mu_n(\text{there are at least } k \text{ submaps isomorphic to } P) = 1.$$

Definition. *A class of maps has* large representativity *if for every $k > 0$,*

$$\lim_{n\to\infty} \mu_n(\text{the representativity is at least } k) = 1.$$

When both of these properties hold, we have a result similar to the result for random circular words used in the proof of Theorem 2.3.

Proposition 5.1. *Let $\mathcal{C}$ be a rich class of maps with large representativity, P be a fixed planar map occurring as a submap of some map in $\mathcal{C}$, and k be a non-negative integer. Then*

$$\mu(\forall a_1, \ldots, a_k\ \exists \text{ a submap isomorphic to } P \text{ not containing } a_1, \ldots, a_k) = 1.$$

Here $a_1, \ldots, a_k$ range over vertices, edges, and faces.

Proof. We must overcome a difficulty not present for circular words: there is no *a priori* upper bound on the number of submaps that contain a and are isomorphic to P.

Find a planar map P' with a distinguished face so that P is a submap of P'; all vertices, edges, and faces of P are *internal* in P' (*i.e.*, none of them are adjacent to the distinguished face of P'); and P' appears as a submap in the class. That such a P' exists is easy to see. In maps of $\mathcal{C}$ consider neighborhoods of submaps isomorphic to P. When the maps have sufficiently large representativity, these neighborhoods will be planar.

We claim that if $a_1, \ldots, a_k$ are vertices, edges, and faces of M, the number of submaps of M isomorphic to P' containing at least one of $a_1, \ldots, a_k$ as an internal vertex, edge, or face is bounded while, by richness, the number of submaps isomorphic to P' is unbounded. Suppose, for example, that a_1 is a vertex with degree d. If an isomorphic copy of P' contains a_1 as an internal vertex, the vertex in P' that maps onto a_1 must also have degree d. (This is not the case if the vertex is not internal.) Let r be the number of internal vertices of P' with degree d. Now map one of these vertices v to a_1, choose an orientation for P', and then map the faces adjacent v to the faces adjacent to a_1 in the same cyclic order. For each v, there are $2d$ ways to do this. This mapping can be extended to an embedding of P' as a submap of M in at most one way. Thus, there are at most $2rd$ isomorphic copies of P' that contain a_1. Similar arguments work for edges and faces.

We conclude that when n is sufficiently large, there will be a submap of M isomorphic to P' not containing $a_1, \ldots, a_k$ as internal vertices, edges, or faces. Within P' is a copy of P not containing $a_1, \ldots, a_k$. ∎

We are now ready to prove the main theorem.

Theorem 5.2. *Let $\mathcal{C}$ be a rich class of maps with large representativity. Then the class of full cross representations of maps in $\mathcal{C}$ has a first-order 0–1 law. Moreover, if $\mathcal{C}'$ is another rich class of maps with large representativity, and $\mathcal{C}$ and $\mathcal{C}'$ have the same planar submaps, then the almost certain sentences for $\mathcal{C}$ and $\mathcal{C}'$ are the same.*

Proof. We need to make some modifications to the proof of Theorem 2.3.

Fix an integer $m \geq 0$. We will consider maps of representativity at least $2^{m+2}+8$. Since $\mathcal{C}$ has large representativity, almost all maps in $\mathcal{C}$ have this property.

Take an element a in the full cross representation $\mathfrak{A}$ of such a map and pick r such that $1 < r \leq 2^{m+1}+2$. If $a \in A_c$, we suppose also that r is even; if $a \in A_v \cup A_f$, we suppose r is odd. By Proposition 4.7, $B^{\mathrm{A}}(a; r)$ is planar.

So far the situation looks as it did for circular words, but there is one crucial difference: there may be (depending on which maps are in $\mathcal{C}$) infinitely many balls of the form $B^{\mathrm{A}}(a; r)$. The reason is that degrees of vertices are not bounded.

Let us think of a ball $B^{\mathrm{A}}(a; r)$ as a structure over the vocabulary consisting of the full cross representation relation symbols plus a constant interpreting a. On structures over this vocabulary $\equiv_{m+1}$ is an equivalence relation of finite index. From every $\equiv_{m+1}$-class that contains one of the balls $\langle B^{\mathrm{A}}(a; r), a\rangle$ pick a representative ball. Let $\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_k$ be a list of all such representatives. Although the region covered by $\mathfrak{B}_i$ is planar, it is not necessarily a map because it may be missing a face, or even several faces since balls need not be simply connected. If $\mathfrak{B}_i$ is missing some faces, find a structure of representativity at least $2^{m+2}+8$ in $\mathcal{C}$ containing it and add additional faces, edges and vertices to $\mathfrak{B}_i$ to form a planar submap P_i of $\mathfrak{A}$. By Proposition 5.1, almost every map in $\mathcal{C}$ has the property that for every sequence of elements $a_1, a_2, \ldots, a_{m-1}$, there is a submap isomorphic to P_i not containing any of them.

Let $\mathcal{E}'$ the class of full cross representations such that for each $\mathfrak{B}_i$, as above, and each sequence of elements $a_1, a_2, \ldots, a_{m-1}$, there is a substructure isomorphic to $\mathfrak{B}_i$ not containing $a_1, a_2, \ldots, a_{m-1}$. We have that $\mathcal{E}'$ has asymptotic probability 1. We will show that Duplicator has a winning strategy in the m-round Ehrenfeucht-Fraïssé game played on every pair of structures from $\mathcal{E}'$.

Take $\mathfrak{A}, \mathfrak{B} \in \mathcal{E}'$ and suppose that at the beginning of the k-th round of the game, the previously picked elements from $\mathfrak{A}$ are $a_1, a_2, \ldots, a_{k-1}$, and the previously picked elements from $\mathfrak{B}$ are $b_1, b_2, \ldots, b_{k-1}$.

Duplicator constructs a digraph $\mathcal{D}$ on the vertex set $\{1, 2, \ldots, m\}$ as the game progresses. Whenever $i < j$ and a_j is chosen so that $d^{\mathrm{A}}(a_i, a_j) \leq 2^{m-j+1}$ or b_j is chosen so that $d^{\mathrm{B}}(b_i, b_j) \leq 2^{m-j+1}$, a directed edge (j, i) is added to $\mathcal{D}$. (Notice that the distance used here is twice what it was for circular words.) Again, j is an *anchor* for k whenever there is a path from k to j in $\mathcal{D}$ and j has out-degree 0. As in the proof of Theorem 2.3, we can show by induction that every $k \leq m$ has a unique anchor.

Suppose that Spoiler now picks an element, say a_k, from $\mathfrak{A}$ (The argument is symmetrical if he chooses b_k from $\mathfrak{B}$.) Duplicator finds the anchor j for k and sets $r_k = 2^{m-k+1} + 2$ if $a_k \in A_e$ or $r = 2^{m-k+1} + 1$ if $a_k \in A_v \cup A_f$.

Suppose $j = k$. There is an i such that $\mathfrak{B}_i \equiv_{m+1} B^{\mathrm{A}}(a_k; r)$. Since $\mathfrak{B} \in \mathcal{E}'$, Duplicator may choose b_k so that $B^{\mathrm{B}}(b_k; r)$ is isomorphic to $\mathfrak{B}_i$ and does not contain $b_1, b_2, \ldots, b_{k-1}$ by Proposition 5.1.

If $j \neq k$, then elements a_j and b_j were chosen so that $\langle B^{\mathrm{A}}(a_j; r_j), a_j\rangle \equiv_{m+1} \langle B^{\mathrm{B}}(b_j; r_j), b_j\rangle$. Thus, Duplicator has a winning strategy in the $(m+1)$-move Ehrenfeucht-Fraïssé game played on this pair of structures. (Note that we use both directions of Theorem 2.2 in the proof of this 0–1 law.) Duplicator uses this strategy in the game on $\mathfrak{A}$ and $\mathfrak{B}$ whenever Spoiler plays an element a_l (or b_l) where j is an anchor for l. By the time Spoiler plays a_k several other elements with the same anchor may have been played, but certainly no more than $m-1$ elements, so the strategy is good for picking b_k, provided that we can show that a_k is in $B^{\mathrm{A}}(a_j; r_j)$.

There is a path from k to j in $\mathcal{D}$. Corresponding to each edge on this path is a pair of elements a_p and a_q where $p < q$, so either $d^{\mathrm{A}}(a_p, a_q) \leq 2^{m-q+1}$ or $d^{\mathrm{B}}(b_p, b_q) \leq 2^{m-q+1}$. Now Duplicator either chose a_q in response to Spoiler's choice of b_q or b_q in response to Spoiler's choice of a_q. When she made this move, it was at most the $(q-1)$-st move of an $(m+1)$-move game played on $\langle B^{\mathrm{A}}(a_j; r_j), a_j\rangle$ and $\langle B^{\mathrm{B}}(b_j; r_j), b_j\rangle$. She plays a winning strategy in this game and there are at least $m-q+2$ moves remaining. We can therefore regard this as a winning position for Duplicator in an $(m-q+2)$-move game and conclude that

$$\langle B^{\mathrm{A}}(a_j; r_j), a_p, a_q\rangle \equiv_{m-q+2} \langle B^{\mathrm{B}}(b_j; r_j), b_p, b_q\rangle.$$

By Proposition 4.1 and Proposition 4.6,

$$d^{\mathrm{A}}(a_p, a_q) = d^{\mathrm{B}}(b_p, b_q) \leq 2^{m-q+1}.$$

Thus, summing over edges in $\mathcal{D}$,

$$d(a_j, a_k) \leq 2^{m-j} + 2^{m-j-1} + \ldots + 2^{m-k+1} < 2^{m-j+1}$$

so a_k is in $B^{\mathrm{A}}(a_j; r_j)$.

Elements related by the full cross relations may be up to distance 2 apart (*e.g.*, elements related by R and T). This is the reason we used balls with radius twice that used for circular words. We wanted to make sure that balls picked during the game have radius greater than 2. The strategy ensures if $d(a_i, a_j) \leq 2$ or $d(b_i, b_j) \leq 2$, there is a k such that a_i, a_j are in $B^{\mathrm{A}}(a_k; 2)$, b_i, b_j are in $B^{\mathrm{B}}(b_k; 2)$, and $\langle B^{\mathrm{A}}(a_k; 2), a_i, a_j\rangle \equiv_2 \langle B^{\mathrm{B}}(b_k; 2), b_i, b_j\rangle$. Therefore, a_i, a_j must satisfy the same relations as b_i, b_j. This is a winning position for Duplicator.

The proof of the second part of the theorem is the same. If $\mathcal{C}$ and $\mathcal{C}'$ have the same planar submaps, and we take $\mathfrak{A}$ from $\mathcal{C}$ and $\mathfrak{B}$ from $\mathcal{C}'$, Duplicator almost certainly has a winning strategy in the m-move Ehrenfeucht-Fraïssé game on $\mathfrak{A}$ and $\mathfrak{B}$. ∎

Corollary 5.3. *The following classes of maps on a surface of type t have a 0–1 law:*

(i) the class of all maps;
(ii) the class of smooth maps;
(iii) the class of 2-connected maps;
(iv) the class of 3-connected maps;
(v) the class of triangular maps;
(vi) the class of 2-connected triangular maps;
(vii) the class of 3-connected triangular maps.

In each of these cases, the set of sentences with asymptotic probability 1 is independent of t.

Proof. In all cases except (iv) and (vii), richness follows from results in section 5 of [3]. For (iv) and (vii), richness follows from Theorem 1 of [8]. Large representativity for (i), (ii), and (iii) follows from [5]. Large representativity for (iv) follows from [5] and [7]. Large representativity for (v) follows from [5] and [20]. Large representativity for (vi) follows from [5] and [21]. Large representativity for (vii) follows from [5] and [22]. The results in these references are all stated for rooted maps. However, by [29] they are valid as stated here. ∎

Remark. For maps on the plane, large representativity is trivial and so only richness needs to be verified for a 0–1 law to hold. Richness has been proved for many classes of planar maps. See [3] and [4] for a discussion. The proofs of richness require that the number of n-edge maps in a class grows like $r^{n(1+o(1))}$ for some $r > 0$. Such a class is said to be *smoothly growing*. All of the smoothly growing classes of planar maps listed in Theorem 1 of [3] and the table in [26] have 0–1 laws. The descriptions of these classes are somewhat technical, so we do not include them here.

6. CONCLUSION

Theorem 5.2 gives 0–1 laws for many classes of maps, but one interesting question is still open. Does the class of planar *graphs* have a 0–1 law? By this we mean that the probability measure is the uniform measure on graph isomorphism types; *i.e.*, graphs are not weighted according to the number of embeddings in the plane. The answer to this question is probably yes, but all efforts to obtain an asymptotic estimate for the number of planar graphs have been unsuccessful. However, proving a 0–1 law may be easier than obtaining an asymptotic estimate. As we have seen, the proof of the 0–1 law requires only richness. (Large representativity is not an issue for planar maps.)

The 0–1 law does hold for 3-connected graphs embeddable in a surface of fixed type, and for 3-connected graphs embeddable as triangular maps in a surface of fixed type. The reason is that by [7] almost all the maps in these two classes have unique embeddings in the surface. Thus, the 0–1 laws follow directly from Corollary 5.3 (iv) and (vii).

It should be possible to derive 0–1 laws for other classes of maps using Theorem 5.2. Some richness results are known for various kinds of maps counted by vertices and faces simultaneously [6], but large representativity results are not available.

As we remarked in the introduction, the 0–1 law for relational structures has been used in expected time analyses of database query optimizations [1]. This analysis assumed that all databases (or structures over a given vocabulary) are equally likely. This is certainly not the case in practice. It would be interesting to do an expected time analysis of query optimization where databases are random maps on a surface. This may be a more realistic model of certain kinds of databases. The 0–1 law for relational structures is very different from the 0–1 laws proved in this paper. For example, the 0–1 law for relational structures does not use a property such as richness; indeed, a random digraph almost certainly has diameter two, which contradicts any kind of richness property. An expected time analysis for random maps of the query optimization technique described in [1] would be more difficult than the analysis for random structures.

Finally, we remark that the proof of Theorem 5.2, based on game strategies and keeping track of *anchors*, is quite general. It should have other applications, and may simplify some proofs of 0–1 laws already in the literature.

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