The Effects of Elementary Row Operations on $\det(A)$

The idea is to turn things around somewhat from the book. We begin with the study of elementary row operations (Section 2.2 in text) and then move backwards to prove the unproved Theorem 2.1.1. Unlike the text, this approach does not leave a major theorem unproved.

Let A be an $n \times n$ matrix. Recall that M_{ij} is A with row i and column j removed. Define $M_{(ik)(jl)}$ to be A with rows i and k and columns k and l removed. Given a matrix B define N_{ij} and $N_{(ik)(jl)}$ similarly.

Lemma 1 We have

$$\det(A) = \sum a_{1i}a_{2j}(-1)^{1+i+j+\chi(i>j)} \det(M_{(12)(ij)}),$$

where the sum is over all i and j between 1 and n for which $i \neq j$ and

$$\chi(\textbf{statement}) = \begin{cases} 1, & \text{if } \textbf{statement} \text{ is true,} \\ 0, & \text{if } \textbf{statement} \text{ is false.} \end{cases}$$

Proof This is easy to verify for 2×2 matrices, so assume $n > 2$. By definition

$$\det(A) = \sum_i a_{1i}(-1)^{1+i} \det(M_{1i}).$$

I claim that

$$\det(M_{1i}) = \sum_j a_{2j}(-1)^{j+\chi(i>j)} \det(M_{(12)(ij)}),$$

To see this, note that when $i > j$, column j of A with the first entry removed is column j of M_{1i}, but, when $i < j$, it is column $j-1$ of $M_{1,i}$. This completes the proof.

Lemma 2 Interchanging two rows of A changes the sign of the determinant.

Proof The 2×2 case is easy, so assume $n > 2$. Suppose the rows are i and $j > i$. Let A be the original matrix and B the matrix with the rows swapped. If $i > 1$ we can use induction on n:

$$\det(B) = \sum_k b_{1k}(-1)^{1+k} \det(N_{1k}).$$

Since $i > 1$, $b_{1k} = a_{1k}$ and N_{1k} is M_{1k} with rows $i-1$ and $j-1$ interchanged. By induction $\det(N_{1k}) = -\det(M_{1k})$. Thus

$$\det(B) = -\sum_k a_{1k}(-1)^{1+k} \det(M_{1k}) = -\det(A).$$

Now suppose $i = 1$ and $j = 2$. Let B be the matrix with rows 1 and 2 switched. Applying Lemma 1 to B we have

$$\begin{aligned} \det(B) &= \sum b_{1i}b_{2j}(-1)^{1+i+j+\chi(i>j)} \det(N_{(12)(ij)}) \\ &= \sum a_{2i}a_{1j}(-1)^{1+i+j+\chi(i>j)} \det(M_{(12)(ij)}) \\ &= -\sum a_{1j}a_{2i}(-1)^{1+i+j+\chi(j>i)} \det(M_{(12)(ij)}), \end{aligned}$$

since $(-1)^{\chi(i>j)}$ and $(-1)^{\chi(j>i)}$ have opposite signs. (This is because one of them is 1 and the other is 0.)

Finally, suppose $i = 1$ and $j > 2$ We can interchange rows 1 and j by doing $2j - 3$ interchanges of adjacent rows:

$$[1, 2],\ [2, 3], \ldots,\ [j - 1, j],\ [j - 2, j - 1], \ldots,\ [1, 2],$$

where $[k, m]$ means swap rows k and m. Thus there are $2j - 3$ sign changes and so we are done with the proof.

Lemma 3 Multiplying a row by a constant c multiplies the determinant by c.

Proof Let the new matrix be B. Suppose row k is multiplied by c. If $k = 1$, $N_{1i} = M_{1i}$ and so

$$\det(B) = \sum c a_{1i} A_{1i} = c \sum a_{1i} A_{1i} = \det(A).$$

If $k > 1$, we use induction on n. In this case N_{1i} is M_{1i} with row $k - 1$ multiplied by c and so $\det(N_{1i}) = c \det(M_{1i})$ by induction. Thus

$$\det(B) = \sum b_{1i}(-1)^{1+i} c \det(M_{1i}) = c \sum a_{1i} A_{1i} = \det(A).$$

We are done with the proof.

Lemma 4 Adding c times row i to row j does not change the determinant.

Proof We can swap rows so that row i becomes the first row and row j becomes the second row. This requires 0, 1 or 2 swaps depending on i and j. Now do the addition, and swap the rows back to where they belong. That requires the same number of swaps as before, so the sign changes cancel by Lemma 2. Thus we can assume $i = 1$ and $j = 2$.

Let the matrix after addition be B. Note that $N_{(12)(ij)} = M_{(12)(ij)}$ By Lemma 1,

$$\begin{aligned}
\det(B) &= \sum b_{1i} b_{2j} (-1)^{1+i+j+\chi(i>j)} \det(N_{(12)(ij)}) \\
&= \sum a_{1i}(a_{2j} + c a_{1j})(-1)^{1+i+j+\chi(i>j)} \det(M_{(12)(ij)}) \\
&= \det(A) + c \sum a_{1i} a_{1j} (-1)^{1+i+j+\chi(i>j)} \det(M_{(12)(ij)}).
\end{aligned}$$

Look at the last sum. Since i and j are just indices of summation, we can replace them with whatever we want. Replace i with j and j with i to obtain

$$\sum a_{1i} a_{1j} (-1)^{1+i+j+\chi(i>j)} \det(M_{(12)(ij)}) = \sum a_{1j} a_{1i} (-1)^{1+j+i+\chi(j>i)} \det(M_{(12)(ji)}).$$

Since $M_{(12)(ij)} = M_{(12)(ji)}$ and $(-1)^{\chi(i>j)}$ and $(-1)^{\chi(j>i)}$ have opposite signs, it follows that the sum on the right is the negative of the sum on the left. Since the only number that equals its negative is zero, the sums are zero and the proof is complete.

We can now prove our main result:

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Theorem 1 If $E_1, \ldots, E_k$ are elementary $n \times n$ matrices and A is an $n \times n$ matrix, then

$$\det(E_1 \cdots E_k A) = \det(E_1) \cdots \det(E_k) \det(A).$$

Proof I claim it suffices to prove that $\det(EB) = \det(E)\det(B)$ whenever E is an elementary $n \times n$ matrix and B is an $n \times n$ matrix. This is proved by induction on k: The case $k = 1$ is $\det(EB) = \det(E)\det(B)$. For $k > 1$

$$\begin{aligned} \det(E_1 \cdots E_k A) &= \det(E_1)\det(E_2 \cdots E_k A) && \text{with } B = E_2 \cdots E_k A, \\ &= \det(E_1) \cdots \det(E_k)\det(A) && \text{by induction.} \end{aligned}$$

To prove $\det(EB) = \det(E)\det(B)$, we deal with the three types of elementary matrices separately.

I. Let E be the elementary matrix that interchanges two rows. By Lemma 2, $\det(EB) = -\det(B)$. Using the definition of det, you should be able to show that $\det(E) = -1$ and so $-\det(B) = \det(E)\det(B)$.

II. Let E be the elementary matrix that multiplies row i by c. Using Lemma 3 and verifying that $\det(E) = c$, you should be able to proceed as in Case I.

III. Let E be the elementary matrix that multiplies row i by c and adds it to row c Using Lemma 4 and verifying that $\det(E) = 1$, you should be able to proceed as in Case I.

This completes the proof.

Lemma 5 If A is an $n \times n$ matrix, then

(a) $\det(A) = 0$ if and only if A is singular and

(b) $\det(A^T) = \det(A)$.

Proof Using elementary row operations, we can convert a matrix to reduced row echelon form. Multiplying by the inverses of the elementary matrices, which are again elementary matrices, we see that any $n \times n$ matrix A can be written in the form

$$A = E_1 \cdots E_k R, \quad \text{where } R \text{ is reduced row echelon.}$$

By Theorem 1, $\det(A) = \det(E_1) \cdots \det(E_k)\det(R)$. Either R is the identity matrix and you should be able to show that $\det R = 1$, or R has a row that is all zeroes and you should be able to show that $\det(R) = 0$.

We know from our study of equations that A is singular if and only if R has a row of zeroes. This proves (a). To prove (b), we consider two cases.

- Suppose A is singular. Then A^T must also be singular. (If it were not singular, $(A^T)^{-1}$ would exist and so $((A^T)^{-1})^T A = (A^T(A^T)^{-1})^T = I^T = I$ and so A^{-1} would exist.) Thus both A and A^T have determinant zero.
- If A is nonsingular, then as noted earlier in the proof, $A = E_1 \cdots E_k$ and $\det(A) = \det(E_1) \cdots \det(E_k)$ by Theorem 1. Note that $A^T = E_k \cdots E_1$. Since the transpose of an elementary matrix is elementary, $\det(A^T) = \det(E_k^T) \cdots \det(E_1^T)$ by Theorem 1.

It is left for you to verify that it follows from the definition of determinants that, if E is an elementary matrix, then $\det(E^T) = \det(E)$. This completes the proof.

We can now prove what the text does not:

Theorem 2.1.1 If A is an $n \times n$ matrix with $n \geq 2$, then $\det(A)$ can be expressed as a cofactor expansion using any row or column of A.

Proof Suppose we want to expand about row k. Perform a series of interchanges so that row k becomes the first row and the other rows are in order. This can be done by interchanging k and $k-1$, then $k-1$ and $j-1$, and so on for a total of $j-1$ interchanges. By Lemma 2 the determinant is multiplied by $(-1)^{k-1}$. Call the new matrix B and note that $\det(B) = (-1)^{k-1}\det(A)$. Expand B using the definition of a determinant. The matrix obtained by deleting row 1 and column j of B is the same as the matrix obtained by deleting k and column j of A. Hence

$$B_{1j} = (-1)^{1+j}\det(M_{kj}) = (-1)^{k+1}(-1)^{k+j}\det(M_{kj}) = (-1)^{k+1}\det(A_{kj}).$$

Thus, expanding A using row k is the same as expanding B using the first row and multiplying by $(-1)^{k+1}$ in other words, it is $(-1)^{k+1}\det(B)$. Since $(-1)^{k-1}(-1)^{k+1} = +1$, this completes the proof for row expansion.

Suppose we want to expand A about column k. This is the same as expanding A^T about row k, which we know from the previous paragraph equals $\det(A^T)$. Recall that we proved $\det(A^T) = \det(A)$. This completes the proof.

This is enough since now the book's results can all be proved.