This material is a supplement to Appendix G of Stewart. You should read the appendix, except the last section on complex exponentials, *before* this material.

Differentiation and Integration

Suppose we have a function $f(z)$ whose values are complex numbers and whose variable z may also be a complex number. We can define limits and derivatives as Stewart did for real numbers. Just as for real numbers, we say the complex numbers z and w are "close" if $|z-w|$ is small, where $|z-w|$ is the absolute value of a complex number.*

- We say that $\lim_{z\to\alpha} f(z) = L$ if, for every real number $\epsilon > 0$ there is a corresponding real number $\delta > 0$ such that

$$|f(z) - L| < \epsilon \qquad \text{whenever} \qquad 0 < |z-\alpha| < \delta.$$

- The derivative is defined by $f'(\alpha) = \lim\limits_{z\to\alpha} \dfrac{f(z)-f(\alpha)}{z-\alpha}$.

Our variables will usually be real numbers, in which case z and α are real numbers. Nevertheless the value of a function can still be a complex number because our functions contain complex constants; for example, $f(x) = (1+2i)x + 3ix^2$.

Since our definitions are the same, the formulas for the derivative of the sum, product, quotient and composition of functions still hold. Of course, before we can begin to calculate the derivative of a particular function, we have to know how to calculate the function.

What functions can we calculate? Of course, we still have all the functions that we studied with real numbers. So far, all we know how to do with complex numbers is basic arithmetic. Thus we can differentiate a function like $f(x) = \frac{1+ix}{x^2+2i}$ or a function like $g(x) = \sqrt{1+i}\, e^x$ since $f(x)$ involves only the basic arithmetic operations and $g(x)$ involves a (complex) constant times a real function, e^x, that we know how to differentiate. On the other hand, we cannot differentiate a function like e^{ix} because we don't even know how to calculate it.

What about integration? Of course, we still define the indefinite integral of a function to be its antidervatives. We only define the definite integral for functions of a real variable. The function, such as $f(x) = (1+2i)x + i3x^2$, may have complex values but the variable x is only allowed to take on real values. In this case nothing changes:

- The Riemann sum definition of an integral still applies.
- The Fundamental Theorem of Calculus is still true.
- The properties of integrals, including substitution and integration by parts still work.

For example,

$$\int_0^2 \left((1+2i)x + ix^2\right)\,dx \;=\; \left(\frac{(1+2i)x^2}{2} + ix^3\right]_0^2 \;=\; (1+2i)2 + 8i \;=\; 2+12i.$$

* The definitions are nearly copies of Stewart Sections 2.4 and 2.8. We have used z and α instead of x and a to emphasize the fact that they are complex numbers and have called attention to the fact that δ and ϵ are real numbers.

On the other hand, we can't evaluate right now $\int_0^1 (x+i)^{-1}\,dx$. Why is that? We'd expect to write $\int (x+i)^{-1}\,dx = \ln(x+i) + C$ and use the Fundamental Theorem of Calculus, but this has no meaning because we only know how to compute logarithms of positive numbers. Some of you might suggest that we write $\ln|x+i|$ instead of $\ln(x+i)$. This does not work. Since $|x+i| = \sqrt{x^2+1}$, the function $f(x) = \ln|x+i|$ only takes on real values when x is real. Its derivative cannot be the complex number $(x+i)^{-1}$ since $(f(x+h) - f(x))/h$ is real.

Exponential and Trigonometric Functions

How should we define e^{a+bi} where a and b are real numbers? We would like the nice properties of the exponential to still be true. Probably the most basic properties are

$$e^{\alpha+\beta} = e^{\alpha}\,e^{\beta} \qquad \text{and} \qquad (e^{\alpha x})' = \alpha e^{\alpha x}. \tag{1}$$

It turns out that the following definition has these properties.

Definition of complex exponential: $e^{a+bi} = e^a(\cos b + i\sin b) = e^a\cos b + ie^a\sin b$

Two questions that might occur are

- How did you come up with this definition?
- How do you know it has the desired properties?

We consider each of these in turn.

Example (Deriving a formula for e^{a+bi}) In Appendix G Stuart uses Taylor series to come up with a formula for e^{a+bi}. Since you haven't studied Taylor series yet, we take a different approach.

From the first of (1) with $\alpha = a$ and $\beta = b$, e^{a+bi} should equal $e^a\,e^{bi}$. Thus we only need to know how to compute e^{bi} when b is a real number.

Think of b as a variable and write $f(x) = e^{xi} = e^{ix}$. By the second property in (1) with $\alpha = i$, we have $f'(x) = if(x)$ and $f''(x) = if'(x) = i^2 f(x) = -f(x)$. It may not seem like we're getting anywhere, but we are!

Look at the equation $f''(x) = -f(x)$. There's not a complex number in sight, so let's forget about them for a moment. Do you know of any real functions $f(x)$ with $f''(x) = -f(x)$? Yes. Two such functions are $\cos x$ and $\sin x$. In fact,

$$\text{If } f(x) = A\cos x + B\sin x, \text{ then } f''(x) = -x.$$

We need constants (probably complex) so that it's reasonable to let $e^{ix} = A\cos x + B\sin x$. How can we find A and B? When $x = 0$, $e^{ix} = e^0 = 1$. Since

$$A\cos x + B\sin x = A\cos 0 + B\sin 0 = A,$$

we want $A = 1$. We can get B by looking at $(e^{ix})'$ at $x = 0$. You should check that this gives $B = i$. (Remember that we want the derivative of e^{ix} to equal ie^{ix}.) Thus we get

$$\boxed{\textbf{Euler's formula:} \quad e^{ix} = \cos x + i \sin x}$$

Putting it all together we finally have our definition for e^{a+bi}. ∎

We still need to verify that our definition for e^z satifies (1). The verification that $e^{\alpha+\beta} = e^\alpha e^\beta$ is left as an exercise. We will prove that $(e^z)' = e^z$ for complex numbers. Then, by the Chain Rule, $(e^{\alpha x})' = (e^{\alpha x})(\alpha x)' = \alpha e^{\alpha x}$, which is what we wanted to prove.

Example (A proof that $(e^z)' = e^z$) By the definition of derivative and $e^{\alpha+\beta} = e^\alpha e^\beta$ with $\alpha = z$ and $\beta = w$, we have

$$(e^z)' = \lim_{w\to 0} \frac{e^{z+w} - e^z}{w} = \lim_{w\to 0} \frac{e^z e^w - e^z}{w} = e^z \lim_{w\to 0} \frac{e^w - 1}{w}.$$

Let $w = x + iy$ where x and y are small real numbers. Then

$$\frac{e^w - 1}{w} = \frac{e^x \cos y + ie^x \sin y - 1}{x + iy}$$

Since x and y are small, we can use linear approximations* for e^x, $\cos y$ and $\sin y$, namely $1 + x$, 1 and y. (The approximation 1 comes from $(\cos y)' = 0$ at $y = 0$.) Thus $(e^w - 1)/w$ is approximately

$$\frac{(1 + x)(1) + i(1 + x)y - 1}{x + iy} = \frac{x + iy + ixy}{x + iy} = 1 + \frac{ixy}{x + iy}.$$

When x and y are very small, their product is much smaller than either of them. Thus $\lim_{w\to 0} \frac{ixy}{x+iy} = 0$ and so $\lim_{w\to 0}(e^w - 1)/w = 1$. This shows that $(e^z)' = e^z$. ∎

Finding Euler's formula and checking that it did what we want was a bit of work. Now that we have Euler's formula, it's easy to get formulas for the trig functions in terms of the exponential. Look at Euler's formula with x replaced by $-x$:

$$e^{-ix} = \cos(-x) + i \sin(-x) = \cos x - i \sin x.$$

We now have two equations in $\cos x$ and $\sin x$, namely

$$\cos x + i \sin x = e^{ix}$$

$$\cos x - i \sin x = e^{-ix}.$$

* Linear approximations are discussed in Section 3.11 of Stewart.

Adding and dividing by 2 gives us $\cos x$ whereas subtracting and dividing by $2i$ gives us $\sin x$:

Exponential form of sine and cosine: $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

Setting $x = \alpha = a + bi$ gives formulas for the sine and cosine of complex numbers. We can do a variety of things with these formula. Here are some we will not pursue:

- Since the other trig functions are rational functions of sine and cosine, this gives us formulas for all the trig functions.
- Identities such as $\cos^2(\alpha) + \sin^2(\alpha) = 1$ can be verified for complex numbers.
- The hyperbolic and trig functions are related: $\cos x = \cosh(ix)$ and $i \sin x = \sinh(ix)$.

Integrating Products of Sine, Cosine and Exponential

In Section 7.1 problems like $\int e^x \cos x \, dx$ were done using integration by parts twice. In Section 7.2 products of sines and cosines were integrated using trig identities. There are easier ways them now that we have complex numbers. Some examples will make this clearer.

Example (Avoiding integration by parts sometimes) Let's integrate $e^{2x} \sin x$. Using the formula for sine and integrating we have

$$\begin{aligned}
\int e^{2x} \sin x \, dx &= \frac{1}{2i} \int e^{2x}(e^{ix} - e^{-ix})\, dx = \frac{1}{2i} \int (e^{(2+i)x} - e^{(2-i)x})\, dx \\
&= \frac{1}{2i} \left(\frac{e^{(2+i)x}}{2+i} - \frac{e^{(2-i)x}}{2-i} \right) + C \\
&= \frac{-ie^{2x}}{2} \left(\frac{e^{ix}(2-i)}{5} - \frac{e^{-ix}(2+i)}{5} \right) + C \\
&= \frac{e^{2x}}{10} \Big((1-2i)(\cos x + i \sin x) + (1+2i)(\cos x - i \sin x) \Big) + C,
\end{aligned}$$

where we used $\cos(-x) = \cos x$ and $\sin(-x) = \sin x$ on the last line. Collecting terms together, we finally get

$$\int e^{2x} \cos x \, dx = \frac{e^{2x}(\cos x + 2 \sin x)}{5} + C.$$

This was done in Stewart using integration by parts twice. ∎

Example (Products of sines and cosines) Let's integrate $8\cos(3x)\sin^2 x$. One way to do this is to automatically reduce the product to a sum of sines and cosines using our formulas for them:

$$\begin{aligned} 8\cos(3x)\sin^2 x &= 8\left(\frac{e^{3ix}+e^{-3ix}}{2}\right)\left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2 \\ &= \frac{e^{5ix}-2e^{3ix}+e^{ix}+e^{-ix}-2e^{-3ix}+e^{-5ix}}{i^2} \\ &= -(e^{5ix}+e^{-5ix})+2(e^{3ix}+e^{-3ix})-(e^{ix}+e^{-ix}) \\ &= -2\cos(5x)+4\cos(3x)-2\cos x. \end{aligned}$$

This function is easily integrated. Another approach is to integrate the second line of the previous result as follows

$$\begin{aligned} \int 8\cos(3x)\sin^2 x\,dx &= \int -(e^{5ix}-2e^{3ix}+e^{ix}+e^{-ix}-2e^{-3ix}+e^{-5ix})\,dx \\ &= \frac{-e^{5ix}}{5i}+\frac{2e^{3ix}}{3i}-\frac{e^{ix}}{i}+\frac{e^{-ix}}{i}-\frac{2e^{-3ix}}{3i}+\frac{e^{-5ix}}{5i}+C. \end{aligned}$$

Next use Euler's formula and combine terms to obtain an answer in terms of sines and cosines. All the imaginary numbers will disappear since the function we were integrating had no imaginary numbers in it. ∎

Logarithms and Inverse Trig Functions

The logarithm function is defined to be the inverse of the exponential function. Since $(e^z)' = e^z$, the argument used at the start of Section 3.8 of Stewart shows that $(\ln z)' = 1/z$. This tells us how to compute the derivative of the logarithm, but we still don't know how to calculate the value of $\ln z$. That's our first goal in this section.

What is $\ln 1$? The answer is 0 if we are limited to real numbers, but with complex numbers it is not so simple. Suppose k is an integer. By Euler's formula,

$$e^{2k\pi i} = \cos(2k\pi) + i\sin(2k\pi) = 1.$$

Since the logarithm is the inverse of the exponential, we could let $\ln 1 = 2k\pi i$ for any integer k. It turns out that $z = 2k\pi i$ are the only solutions to $e^z = 1$. In other words, these are the only possible values for $\ln 1$.

What are the possible values for $\ln(a+bi)$? In other words, what are the solutions to $a+bi = e^z$? Write $z = x+yi$. Then

$$a+bi \;=\; e^z \;=\; e^x(\cos y + i\sin y),$$

which we must solve for x and y. We leave it for you to show that $e^x = |a + bi|$ and $y = \arg(a + bi)$. Thus $x = \ln|a + bi|$. If we write $\alpha = a + bi$, these results become

The logarithm: For $\alpha \neq 0$, possible values of $\ln\alpha$ are $\ln\alpha = \ln|\alpha| + i\arg\alpha$,

where $\ln|\alpha|$ is the standard real logarithm of a positive real number.

Of course, $\arg\alpha$ is not unique: we can add any integer multiple of 2π to it. To actually define a logarithm *function*, we must decide on a *unique choice* among the infinity of values for $\arg\alpha$. This is the same problem that you've already run into with the inverse trig functions in calculus. For example $-\pi/2 < \arcsin t \leq \pi/2$ is the standard convention for arcsin. For ln, **the standard convention is to choose $-\pi < \arg\alpha \leq \pi$.**

Because of the exponential forms of sine and cosine, we can write all the basic trig functions as rational functions of e^{ix}. These can be solved for e^{ix} and logarithms can be taken to get the inverse trig functions. We illustrate with the arctangent.

If $x = \arctan t$ and we let $z = e^{ix}$, then

$$t = \tan x = \frac{\sin x}{\cos x} = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})} = \frac{z - 1/z}{i(z + 1/z)} = \frac{z^2 - 1}{i(z^2 + 1)}.$$

Thus $it(z^2+1) = z^2-1$ and so $z^2 = \frac{1+it}{1-it}$. Recall that $z^2 = (e^{ix})^2 = e^{2ix}$ Taking logarithms and dividing by $2i$, we obtain

$$\arctan t = \frac{1}{2i}\ln\left(\frac{1+it}{1-it}\right)$$

Partial Fractions

How do we know it is always possible to write a rational function as a polynomial plus a sum of partial fractions? It depends on the following result which we will not prove.

Fundamental Theorem of Algebra: Any nonconstant polynomial can be factored as a product of linear factors with complex coefficients; that is, factors of the form $\alpha x+\beta$.

This tells us that we can factor an polynomial of degree n into a product of n linear factors. For example,

- $3x^2 + 2x - 1 = (3x - 1)(x + 1)$ ($n = 2$ here),
- $x^3 - 8 = (x - 2)(x + \alpha)(x + \overline{\alpha})$ where $\alpha = 1 \pm i\sqrt{3}$ ($n = 3$ here),
- $(x^2 + 1)^2 = (x + i)^2(x - i)^2$ ($n = 4$ here).

Thus, if we allow complex numbers, partial fractions can be done with only linear factors. When we only allowed real numbers as coefficients of the factors, we obtained both linear and quadratic factors.

Example (An integral) Using linear factors evaluate

$$\int \frac{4x^3 - 12x + 16\,dx}{(x^2+1)^3}$$

and express the result without complex numbers.

By the theory of partial fractions, there must be constants A_i and B_i so that

$$\frac{4x^3 - 12x + 16}{(x^2+1)^3} = \left(\frac{A_3}{(x+i)^3} + \frac{A_2}{(x+i)^2} + \frac{A_1}{x+i}\right) + \left(\frac{B_3}{(x-i)^3} + \frac{B_2}{(x-i)^2} + \frac{B_1}{x-i}\right).$$

The constants can be found by the method of Stewart's Section 7.4. It turns out that $A_1 = 3i$, $A_2 = -3$, $A_3 = 2 - 2i$ and $B_i = \overline{A_i}$. Thus the integral is

$$\left(\frac{-1+i}{(x+i)^2} + \frac{-1-i}{(x-i)^2}\right) + \left(\frac{3}{x+i} + \frac{3}{x-i}\right) + \Big(3i\ln(x+i) - 3i\ln(x-i)\Big) + C$$

$$= \frac{1+4x-2x^2}{(x^2+1)^2} + \frac{6x}{x^2+1} + 6\arctan x + C,$$

where we used

$$\frac{-1+i}{(x+i)^2} + \frac{-1-i}{(x-i)^2} = \frac{(-1+i)(x-i)^2 - (1+i)(x+i)^2}{(x^2+1)^2} = \frac{-2x^2+4x+1}{(x^2+1)^2},$$

$$\frac{3}{x+i} + \frac{3}{x-i} = \frac{3(x-i)+3(x+i)}{x^2+1} = \frac{6x}{x^2+1}.$$

and the fact that $i\ln(x+i) - i\ln(x-i)$ can be replaced by $2\arctan x$. The fact about the arctangent is left to the exercises. ∎

The partial fraction constants can be found by the method in the text. There are easier methods.

You can save a lot of time when there are no repeated factors in the denominator. We'll tell you the general principle and then do some specific examples. Suppose that

$$x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = (x+\alpha_1)\cdots(x+\alpha_n)$$

where $\alpha_1, \ldots, \alpha_n$ are all distinct. Suppose also that the degree of $p(x)$ is less than n Then

$$\frac{p(x)}{x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n} = \frac{C_1}{x+\alpha_1} + \cdots + \frac{C_n}{x+\alpha_n}, \tag{2}$$

where the constants $C_1, \ldots, C_n$ need to be determined to find the partial fraction expansion. Multiply both sides of (2) by $x + \alpha_i$ and then set $x = -\alpha_i$. The left side is some number. On the right side, we are left with only C_i because all the other terms have a factor of $x + \alpha_i$ which is 0 when $x = -\alpha_i$.

Now for some illustrations.

Example (Partial fractions with no repeated factors. I) Let's expand $\frac{x^2+2}{(x-1)(x+2)(x+3)}$ by partial fractions.

$$\frac{x^2+2}{(x-1)(x+2)(x+3)} \;=\; \frac{C_1}{x-1}+\frac{C_2}{x+2}+\frac{C_3}{x+3}.$$

Multiply by $x-1$:

$$\frac{x^2+2}{(x+2)(x+3)} \;=\; C_1+\frac{C_2(x-1)}{x+2}+\frac{C_3(x-1)}{x+3}.$$

Set $x=1$:

$$\frac{1+2}{(1+2)(1+3)} \;=\; C_1$$

and so $C_1 = 1/4$. Similarly,

$$C_2 \;=\; \left.\frac{x^2+2}{(x-1)(x+3)}\right]_{x=-2} = \frac{4+2}{(-3)1} \;=\; -2$$

and

$$C_3 \;=\; \left.\frac{x^2+2}{(x-1)(x+2)}\right]_{x=-3} = \frac{9+2}{(-4)(-1)} \;=\; 11/4. \quad \blacksquare$$

Example (Partial fractions with no repeated factors. II) Let's expand $\frac{x+1}{x^3+x}$. We have

$$\frac{x+1}{x(x^3+x)} \;=\; \frac{x+1}{x(x-i)(x+i)} \;=\; \frac{C_1}{x}+\frac{C_2}{x-i}+\frac{C_3}{x+i}.$$

Since $x = x-0$,

$$C_1 \;=\; \frac{1}{(-i)i} = 1.$$

Also

$$C_2 \;=\; \frac{i+1}{i(2i)} \;=\; \frac{-1-i}{2} \qquad C_3 \;=\; \frac{-i+1}{(-i)(-2i)} \;=\; \frac{-1+i}{2}.$$

You should fill in the details. ∎

Notice that $C_3 = \overline{C_2}$ in the previous example. This often happens and can save us work. When and why? Suppose $p(x)$ and $q(x)$ are polynomials with real coefficients and think of x as a real number. Suppose $(x+\alpha)^k$ is a factor of $q(x)$ where α not a real number. Then $(x+\overline{\alpha})^k$ is also a factor:

(a) Since $(x+\alpha)^k$ is a factor of $q(x)$, $q(x) \;=\; (x+\alpha)^k r(x)$ for some polynomial $r(x)$.
(b) With complex conjugates, $\overline{q(x)} \;=\; (x+\overline{\alpha})^k \overline{r(x)}$.
(c) Since $q(x)$ has real coefficients, $\overline{q(x)} = q(x)$ and so by (b), $(x+\overline{\alpha})^k$ is a factor of $q(x)$.

Now write out the partial fraction expansion for $p(x)/q(x)$,

$$\frac{p(x)}{q(x)} = \frac{A}{(x+\alpha)^k} + \frac{B}{(x+\overline{\alpha})^k} + \cdots,$$

take complex conjugates, remembering that $\overline{p(x)} = p(x)$,

$$\frac{p(x)}{q(x)} = \frac{\overline{A}}{(x+\overline{\alpha})^k} + \frac{\overline{B}}{(x+\alpha)^k} + \overline{\cdots},$$

and compare to see that $A = \overline{B}$ and $B = \overline{A}$. In summary, it always happens when expanding a rational function that has only real coefficients.

We now do an example where the denominator has repeated factors.

Example (Partial fractions with repeated factors) Let's expand the function

$$f(x) = \frac{4x^3 - 12x + 16}{(x^2+1)^3}$$

that we integrated in an earlier example. By the previous paragraph,

$$\frac{4x^3 - 12x + 16}{(x^2+1)^3} = \left(\frac{A_3}{(x+i)^3} + \frac{A_2}{(x+i)^2} + \frac{A_1}{x+i}\right) + \left(\frac{\overline{A_3}}{(x-i)^3} + \frac{\overline{A_2}}{(x-i)^2} + \frac{\overline{A_1}}{x-i}\right).$$

If we multiply by $(x+i)^3$ and set $x = -i$, we get A_3 in the same way as happened when there were no repeated factors in the denominator. Thus we find $A_3 = 2 - 2i$ and so $B_3 = \overline{A_3} = 2 + 2i$.

How can we get A_1 and A_2? We can't simply multiply by $(x+i)^2$ and set $x = -i$ in an attempt to find A_2 because some terms will have a factor of $x+i$ remaining in their denominator and so will become infinite. So what can we do? There are at least three methods.

- Go back to the Stewart approach, with the values of A_3 and B_3 known, so there are two less unknowns. This is not attractive since what we've been doing is easier than Stewart.
- Subtract the known terms to obtain a function with smaller denominator: Let

  $$h(x) = f(x) - \frac{A_3}{(x+i)^3} - \frac{B_3}{(x-i)^3}.$$

  After cancelling common factors from numerator and denominator, the denominator of $h(x)$ will be $(x^2+1)^2$. In fact, $h(x) = \frac{12}{(x^2+1)^2}$ The method used for finding A_3 and B_3 for $f(x)$ can now be used to find A_2 and B_2 for $h(x)$. Another round of this process gives A_1 and B_1.
- Recall that

  $$\frac{4x^3 - 12x + 16}{(x-i)^3} = A_3 + A_2(x+i) + A_1(x+i)^2 + g(x)(x+i)^3, \tag{3}$$

where $g(x)$ is a rational function with no powers of $x+i$ in the denominator. Notice that the first and second derivatives of $g(x)(x+i)^3$ have at least one factor of $x+i$ and so are zero at $x=-i$.

Differentiate both sides of (3) and again set $x=-i$. This gives A_2. Repeat to obtain A_1. The derivative of the product $(4x^3-12x+6)(x-i)^{-3}$ is

$$(12x^2-12)(x-i)^{-3}-3(4x^3-12x+16)(x-i)^{-4}.$$

The derivative of the right side of (3) evaluated at $x=-i$ is A_2. Thus

$$\begin{aligned} A_2 &= (-12-12)(-2i)^{-3}-3(-4i-12i+16)(-2i)^{-4} \\ &= 24/8i+3(16i-16)/16 \;=\; -3. \end{aligned}$$

Differentiating (3) twice and setting $x=-i$ gives $2A_1$ on the right side. The derivative of the left side is

$$(24x)(x-i)^{-3}-6(12x^2-12)(x-i)^{-4}+12(4x^3-12x+16)(x-i)^{-5},$$

which we leave to you to evaluate at $x=-i$. ■

Exercises

1. The goal of this exercise is to use the definition of a complex exponential to prove that $e^{\alpha+\beta}=e^{\alpha}+e^{\beta}$. Let $\alpha=a+bi$ and $\beta=c+di$.
   (a) Show that, by the definition of a complex exponential, $e^{\alpha+\beta}=e^{a+c}(\cos(b+d)+i\sin(b+d))$.
   (b) State similar formulas for e^{α} and e^{β}.
   (c) Using some algebra and the formulas for the sine and cosine of a sum of angles, conclude that $e^{\alpha+\beta}=e^{\alpha}\,e^{\beta}$.

2. Use the relationship between the sine, cosine and exponential functions to express $\cos^3 x$ as a sum of sines and cosines.

3. Prove that $e^{\pi i}+1=0$. This uses several basic concepts in mathematics (π, e, addition, multiplication, exponentiation and complex numbers) in one compact equation.

4. When discussing integration near the start of these notes, we argued that $f(x)=\ln|x+i|$ could not be an antiderivative of $(x+i)^{-1}$ because $f(x)$ is real valued. Another way to do this is to simply compute $f'(x)$. Do that and check that $f'(x)\neq(x+i)^{-1}$.

5. In the text we learned that $\int dx/x=\ln|x|+C$. Using complex numbers, we can write $\int dx/x=\ln x+C$ because $\ln|x|$ and $\ln x$ differ by a constant when $x<0$. Find the constant.

6. In Box 3 of Appendix G, Stewart states a formula for roots of a complex number $z = re^{i\theta}$ which he derived using De Moivre's Theorem. In this exercise, you do it another way.
   (a) List all possible values for $\ln z$.
   (b) Since $\ln(z^{1/n}) = (1/n)\ln z$, use (a) to list all possible logarithms of $z^{1/n}$.
   (c) Since $z^{1/n} = e^{\ln(z^{1/n})}$, you can use (b) to list all possible values of $z^{1/n}$. Do it.

7. The purpose of this exercise is to derive a formula for $\arcsin t$ like the formula for $\arctan t$. Let $\sin x = t$ and $z = e^{ix}$.
   (a) Derive the formula $z = it \pm \sqrt{1-t^2}$.
   (b) By setting $x = 0$, decide whether $z = it + \sqrt{1-t^2}$ or $z = it - \sqrt{1-t^2}$.
   (c) Complete the derivation of a formula expressing $\arcsin t$ in terms of ln.

8. In this exercise, you use a different method to derive the formula for $\arctan t$ that appears in these notes. We can write $1 + x^2 = (1 - ix)(1 + ix)$. Use this and partial fractions to compute $\int_0^t \frac{1}{1+x^2}\,dx$, which equals $\arctan t$.

9. By using partial fractions, we have

$$\int \frac{dx}{x^2+1} = \frac{1}{2}\int\left(\frac{i}{x+i} - \frac{i}{x-i}\right)dx = \frac{i}{2}\Big(\ln(x+i) - \ln(x-i)\Big) + C$$
$$= \frac{i}{2}\ln\left(\frac{x+i}{x-i}\right) + C.$$

   This is doesn't look like the formula for the arctangent derived in the text even though $\int(x^2+1)^{-1}\,dx = \arctan x + C$. We could show that they agree by differentiating both expressions and showing that the results are equal. Find a more "direct" approach: Use properties of the logarithm and $\arctan 0 = 0$ to show that the two formulas agree.

10. By looking at a right triangle, one can see that $\arctan t = \pi/2 - \arctan(1/t)$ when $t > 0$. By using properties of logarithms, show that our formula for the arctangent satisfies this equation.

11. Find the partial fraction expansion of $\frac{x^3+2}{x(x^2-1)(x^2-4)}$.

12. Find the partial fraction expansion of $\frac{2x+1}{(x-1)^2(x+2)}$.

13. Find the partial fraction expansion of $\frac{x^3+2}{x(x^2+1)(x^2+4)}$.