Section 7.8: Improper Integrals of Rational Functions

Theorem. Let $p(x)$ and $q(x)$ be polynomials with no common factor. Let $a < b$ be real numbers or infinity. The integral $\int_a^b \frac{p(x)}{q(x)}\,dx$ is divergent if either of the following conditions is true. If neither condition is true, the integral either is not improper or is convergent.

(1) There is a c such that $a \le c \le b$ and $q(c) = 0$.

(2) Either a or b is infinite and $\deg(q(x)) - \deg(p(x)) \le 1$.

Here $\deg(p(x))$ is the degree of the polynomial $p(x)$.

Proof: Recall that if a polynomial $r(x)$ has $r(c) = 0$, then $r(x) = (x - c)s(x)$ for some polynomial $s(x)$. Thus, we cannot have $p(c) = q(c) = 0$ because they would have $x - c$ as a common factor.

Suppose from now on that the integral is improper. Split the integral into pieces so that either the upper or lower limit of each integral is a problem, but not both. By definition, if any of these integrals diverges, our integral is divergent. We treat each integral separately, calling its limits a and b.

First suppose that (1) holds. Then either $a = c$ or $b = c$. We do the first case since the second is done the same way. We know that the integrand can be split into partial fractions. Combine all the fractions that have powers of $x - a$ in the denominator and call the result $r(x)/(x - a)^n$. The remaining partial fractions can be integrated with no problem since their denominators do not vanish on the interval of integration. (Remember, we split the interval of integration up in the last paragraph so the only bad point for the integral will be at a.) By replacing $r(x)$ with $-r(x)$ if necessary, we can assume that $r(a) > 0$. Since $r(x)$ is continuous, there is a B with $a < B \le b$ such that $r(x) > r(a)/2$ for $a \le x \le B$. Hence $p(x)/q(x) > r(a)/2(x - a)^n > 0$. By the comparison theorem, the integral from a to B diverges.

Now suppose that (2) holds. We assume that the upper limit is infinite since the lower limit is treated similarly. Factor $x^{\deg(p(x))}$ out of $p(x)$, leaving a polynomial $p^*(1/x)$ in $1/x$ with nonzero constant term, say p_0. Do the same with $q(x)$. By replacing $p(x)$ with $-p(x)$ if necessary, we can assume that $p_0/q_0 > 0$. Then

$$\frac{p(x)}{q(x)} = \frac{p*(1/x)}{q^*(1/x)}\,x^{-d} \quad \text{where} \quad d = \deg(q(x)) - \deg(p(x)) \quad \text{and} \quad \lim_{x\to\infty} \frac{p^*(1/x)}{q^*(1/x)} = \frac{p_0}{q_0}.$$

From the last part of this statement, it follows that

$$2p_0/q_0 \ge p^*(1/x)/q^*(1/x) \ge p_0/2q_0$$

for all sufficiently large x, say $x \ge A$. Thus $(2p_0/q_0)x^{-d} \ge p(x)/q(x) \ge (p_0/2q_0)x^{-d}$. By applying the comparison theorem twice (Do it!), we have divergence of the integral from A to ∞ if and only if $d \le 1$.

This completes the proof.