The error bounds for numerical integration are presented without proof. While it is perhaps unreasonable to prove all of them in an introductory text, one should at least prove the bound for the Trapezoidal Rule since it is a nice application of integration by parts. (The Midpoint Rule is, too — see exercises at the end.) We do that here.

Suppose we want to estimate $\int_a^b f(x)\,dx$ using the Trapezoidal Rule with n intervals. As usual, let $h = \frac{b-a}{n}$ and $x_i = a + ih$. We look at a single interval and integrate by parts twice:

$$\begin{aligned}\int_{x_i}^{x_i+1} f(x)\,dx &= \int_0^h f(t+x_i)\,dt = \Big[(t+A)f(t+x_i)\Big]_0^h - \int_0^h (t+A)f'(t+x_i)\,dt \\ &= \Big[(t+A)f(t+x_i)\Big]_0^h - \left[\left(\frac{(t+A)^2}{2}+B\right)f'(t+x_i)\right]_0^h \\ &\quad + \int_0^h \left(\frac{(t+A)^2}{2}+B\right)f''(t+x_i)\,dt,\end{aligned}$$

where we can choose the constants of integration A and B any way we choose. We want to choose A so that $\Big[(t+A)f(t+x_i)\Big]_0^h$ is the trapezoid area and B so that our error bound will be small.

For A, we want $(h+A)f(h+x_i) - Af(x_i) = (f(x_i)+f(x_{i+1}))h/2$. Since $h+x_i = x_{i+1}$, you should be able the verify that $A = -h/2$ works.

One way we could try make our error bound small is by making

$$\left[\left(\frac{(t+A)^2}{2}+B\right)f'(t+x_i)\right]_0^h$$

equal to zero — if we can. What this will do is push all the error into one place, namely the integral containing f''. Having it all in one place help us get a better bound. Recalling that $A = -h/2$, we have

$$\left[\left(\frac{(t+A)^2}{2}+B\right)f'(t+x_i)\right]_0^h = \left(\frac{(h/2)^2}{2}+B\right)f'(h+x_i) - \left(\frac{(-h/2)^2}{2}+B\right)f'(x_i).$$

Thus we take $B = -h^2/8$ to get zero.

So far we have proved

$$\int_{x_i}^{x_i+1} f(x)\,dx = \frac{h(f(x_i)+f(x_{i+1}))}{2} + \int_0^h \left(\frac{(t-h/2)^2}{2} - \frac{h^2}{8}\right)f''(t+x_i)\,dt$$

Call the difference between the integral and the trapezoid $E_T(i)$. The error in the Trapezoidal Rule equals the sum of these:

$$\begin{aligned}
E_T &= E_T(0) + E_T(1) + \cdots + E_T(n-1) \\
&= \int_0^h \left(\frac{(t-h/2)^2}{2} - h^2/8 \right) f''(t+x_0)\, dt \\
&\qquad + \cdots + \\
&\quad \int_0^h \left(\frac{(t-h/2)^2}{2} - h^2/8 \right) f''(t+x_{n-1})\, dt \\
&= \int_0^h \left(\frac{(t-h/2)^2}{2} - h^2/8 \right) \Big(f''(t+x_0) + \cdots + f''(t+x_{n-1}) \Big)\, dt.
\end{aligned}$$

As in the text, we suppose that $|f''(x)| \le K$ for $a \le x \le b$. Thus

$$\begin{aligned}
|E_T| &= \left| \int_0^h \left(\frac{(t-h/2)^2}{2} - \frac{h^2}{8} \right) \Big(f''(t+x_0) + \cdots + f''(t+x_{n-1}) \Big)\, dt \right| \\
&\le \int_0^h \left| \left(\frac{(t-h/2)^2}{2} - \frac{h^2}{8} \right) \Big(f''(t+x_0) + \cdots + f''(t+x_{n-1}) \Big) \right|\, dt \\
&= \int_0^h \left| \frac{(t-h/2)^2}{2} - \frac{h^2}{8} \right| \Big| f''(t+x_0) + \cdots + f''(t+x_{n-1}) \Big|\, dt \\
&= \int_0^h \left| \frac{(t-h/2)^2}{2} - \frac{h^2}{8} \right| \Big(|f''(t+x_0)| + \cdots + |f''(t+x_{n-1})| \Big)\, dt \\
&\le nK \int_0^h \left| \frac{(t-h/2)^2}{2} - \frac{h^2}{8} \right|\, dt.
\end{aligned}$$

The function $\frac{(t-h/2)^2}{2} - \frac{h^2}{8}$ is a parabola opening upward that is zero at $t = 0$ and $t = h/2$. Thus it is negative for $0 < t < h/2$. Using that fact, we have

$$\begin{aligned}
\int_0^h \left| \frac{(t-h/2)^2}{2} - \frac{h^2}{8} \right| dt &\le \int_0^h \left(\frac{h^2}{8} - \frac{(t-h/2)^2}{2} \right) dt = \left[\frac{h^2 t}{8} - \frac{(t-h/2)^3}{6} \right]_0^h \\
&= \left(\frac{h^3}{8} - \frac{(h/2)^3}{6} + \frac{(-h/2)^3}{6} \right) = \frac{h^3}{12}.
\end{aligned}$$

Putting this all together and using $h = \frac{b-a}{n}$ gives us the error bound in the text:

$$|E_T| \le \frac{nKh^3}{12} = \frac{K(b-a)^3}{12n^2}.$$

Our derivation of the error bound lets us see some weaknesses in it. First, the value of $f''(x)$ can vary from interval to interval. In bounding $|f''(t+x_i)|$ all we need is a bound for $|f''(x)|$ for $x_i < x < x_{i+1}$, which may be much smaller than the bound for $f''(x)$ for $a < x < b$. Second, lets look at what is happening before all the absolute values. We added up the various $f''(t+x_i)$. Since some of these may be negative and some may be positive, there can be cancellation among these terms and so the sum will be much less. What we have is some sort of average of $f''(x)$ multiplied by $\frac{-(b-a)^2}{12n^2}$. Exactly how the average is computed changes as we change n, but it won't change very much from one large value of n to another. Calling this average M, we see that $E_T \approx \frac{-M(b-a)^2}{12n^2}$. As discussed in another supplement to this section of the text, we can use this fact to estimate the error in numerical integration.

Exercises Deriving the Midpoint Rule Error

The derivation of the Midpoint Rule error is similar to that for the Trapezoidal Rule, but each interval has to be broken into two pieces.

1. Using integration by parts twice in each case, derive the identities

$$\begin{aligned}\int_0^{h/2} g(t)\,dt &= \frac{hg(h/2)}{2} - \frac{h^2 g'(h/2)}{8} + \int_0^{h/2} \frac{t^2 g''(t)}{2}\,dt \\ \int_{h/2}^{h} g(t)\,dt &= \frac{hg(h/2)}{2} + \frac{h^2 g'(h/2)}{8} + \int_{h/2}^{h} \frac{(t-h)^2 g''(t)}{2}\,dt.\end{aligned}$$

2. Let E_M be the Midpoint Rule error. Add the results of the previous exercise, replace $g(t)$ with $f(t+x_i)$. Next, by using the ideas in this supplement, derive the following formula, filling in the question marks with the appropriate expressions.

$$E_M = \int_0^{h/2} \frac{t^2}{2}\left(?????\right)\,dt + \int_{h/2}^{h} \frac{???}{2}\left(?????\right)\,dt.$$

3. Conclude that

$$|E_M| \le nK \int_0^{h/2} \frac{t^2}{2}\,dt + nK \int_{h/2}^{h} \frac{???}{2}\,dt,$$

filling the question marks with appropriate expressions. Use this to derive the bound for $|E_M|$ given in the text.